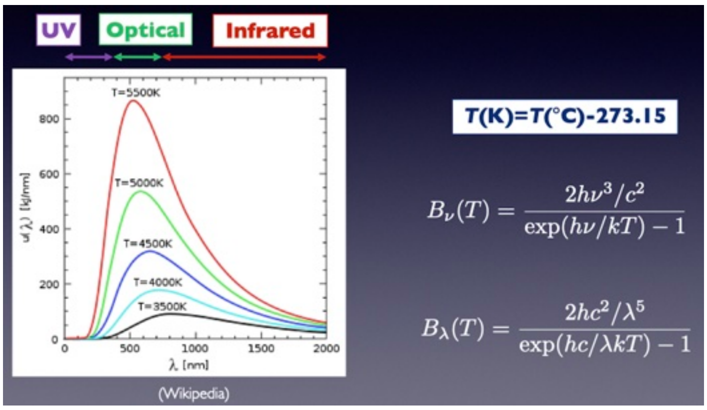
Introduction to Radiative Transfer

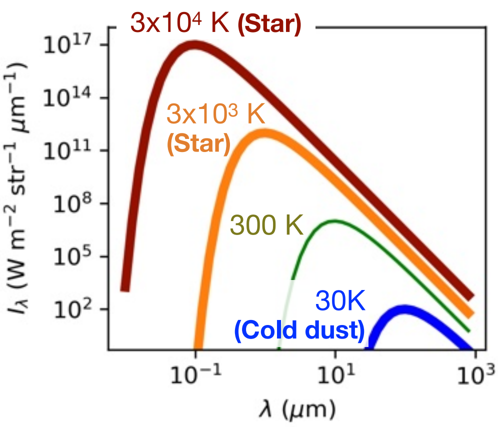
Hiro Takami

References —Lecture notes/slides organized by
C.P. Dullemond, Hiro Hirashita, Seamus Clarke

ASIAA Summer Program, 6/30/2026

Blackbody radiation (100-% Emissivity)

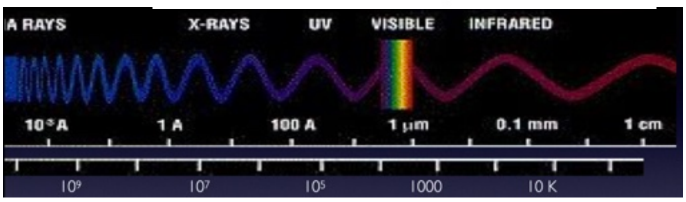




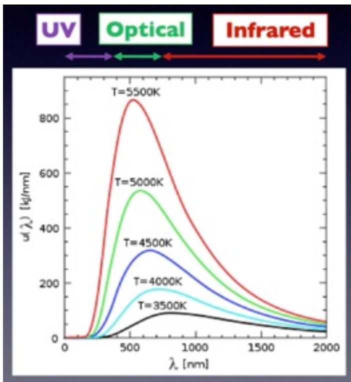
$$\lambda_{\max}[\mu\mathrm{m}] = 2.90 \times 10^3 \;/\; T\ [\mathrm{K}]$$

$$\nu_{\max}[\mathrm{Hz}] = 5.88 \times 10^{10}\ T\ [\mathrm{K}]$$

(Wien's displacement law)



Blackbody radiation (100-% Emissivity)



(Wikipedia)

T(K)=T(°C)-273.15

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$$

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT) - 1}$$

$$B_\nu(T)\, d\nu = B_\lambda(T)\, d\lambda$$

$$\nu B_\nu(T) = \lambda B_\lambda(T)$$

$$\mathbf{B_\nu(T) \neq B_\lambda(T)}$$

$$\lambda_{\max}[\mu\text{m}] = 2.90 \times 10^3 \;/\; T\ [\text{K}]$$

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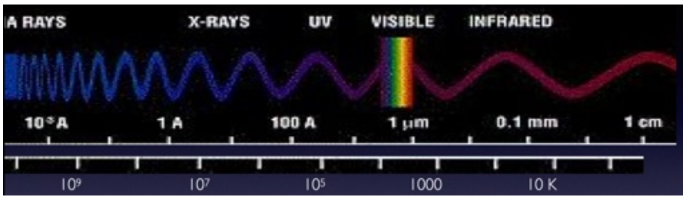
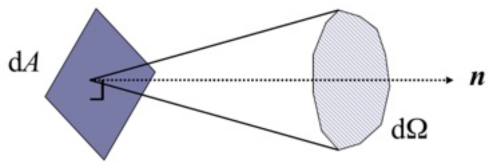
(Wien's displacement law)



Intensity & Flux

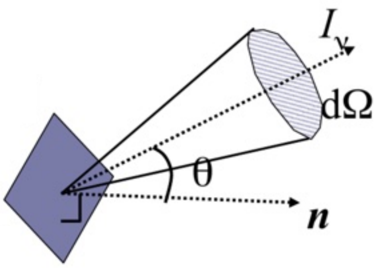


$$\frac{dE}{dt} = \boldsymbol{I_\nu}\, dA\, d\Omega\, d\nu$$

e.g.: W m^{-2} str^{-1} Hz^{-1}

$$= \boldsymbol{I_\lambda}\, dA\, d\Omega\, d\lambda$$

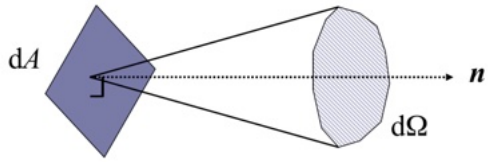
e.g.: W m^{-2} str^{-1} μm^{-1}

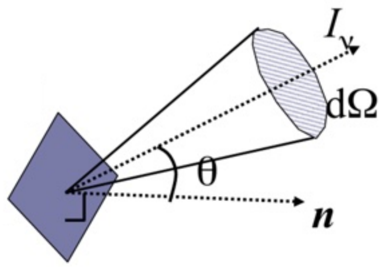


$$\boldsymbol{F_\lambda} = \oint I_\lambda \cos\theta\, d\Omega$$

e.g.: W m^{-2} μm^{-1}

Intensity & Flux





$$\frac{dE}{dt} = \boldsymbol{I_\nu}\ dA\ d\Omega\ d\nu$$

e.g.: W m^{-2} str^{-1} Hz^{-1}

$$= \boldsymbol{I_\lambda}\ dA\ d\Omega\ d\lambda$$

e.g.: W m^{-2} str^{-1} µm^{-1}

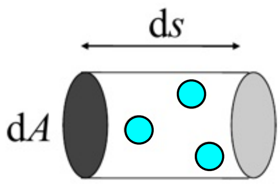
$$I_\nu(T)\ d\nu = I_\lambda(T)\ d\lambda$$

$$\nu I_\nu(T) = \lambda I_\lambda(T)$$

$$\boldsymbol{I_\nu(T) \neq I_\lambda(T)}$$

Radiative Transfer Equation (I)

$$\boxed{\frac{d\boldsymbol{I_\lambda}(s)}{ds} = -\boldsymbol{\alpha_\lambda}(s)\boldsymbol{I_\lambda}(s) + \boldsymbol{j_\lambda}(s)}$$



Extinction coefficient (m^{-1})

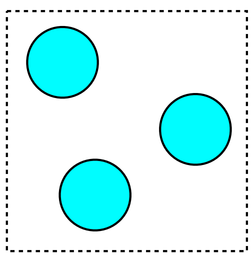
Emissivity (W m^{-3} str^{-1} µm^{-1})

$= \alpha_\lambda$ (absorption) + α_λ (scattering)

$= n\ \sigma_\lambda = n\ (\sigma_{\lambda;\mathrm{abs}} + \sigma_{\lambda;\mathrm{sca}})$

Number density of the particle (m^{-3})

Extinction Cross Section (m^2)

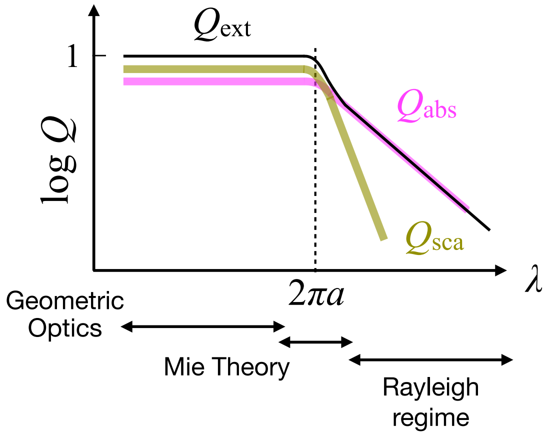


★ Cross Sections for Spherical Particles

$$\sigma_\lambda = Q_{ext}\ \pi a^2$$

$$Q_{ext} = Q_{abs} + Q_{sca}$$

for absorption / \ for scattering



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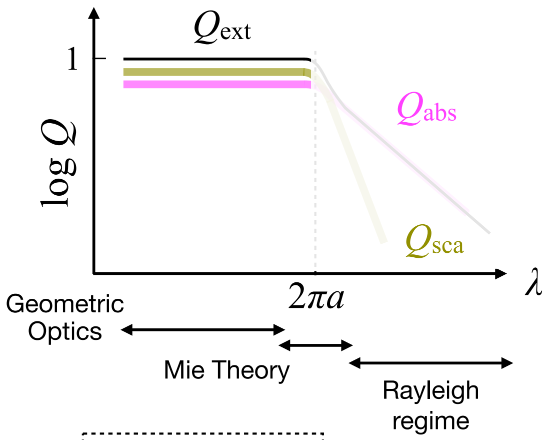
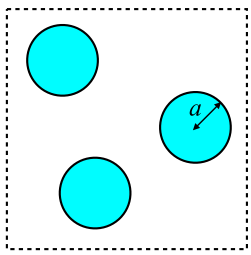


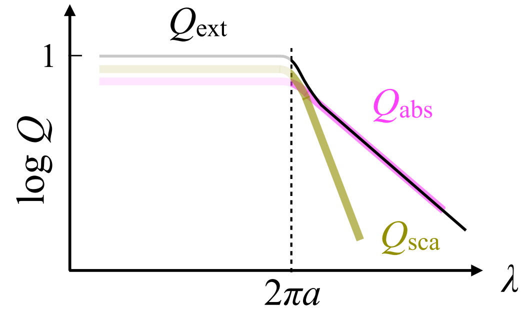
$$\alpha_\lambda = n\sigma_\lambda = n\ \pi a^2$$

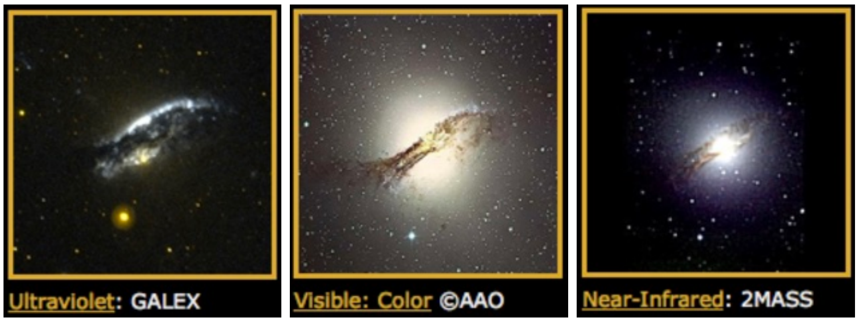


★ Cross Sections for Spherical Particles

$$\sigma_\lambda = Q_{\rm ext}\, \pi a^2$$

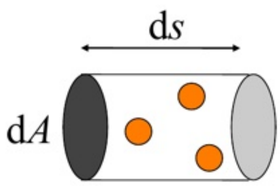
$$Q_{\rm ext} = Q_{\rm abs} + Q_{\rm sca}$$





Radiative Transfer Equation (I)

$$\boxed{\frac{d\boldsymbol{I_\lambda}(s)}{ds} = -\underbrace{\boldsymbol{\alpha_\lambda}(s)}\boldsymbol{I_\lambda}(s) + \underbrace{\boldsymbol{j_\lambda}(s)}}$$



Extinction coefficient (m^{-1})

Emissivity

$= \alpha_\lambda$ (absorption) + α_λ (scattering)

$= n\ \sigma_\lambda = n\ (\sigma_{\lambda;\rm abs} + \sigma_{\lambda;\rm sca})$

$= \rho\ \kappa_\lambda = \rho\ (\kappa_{\lambda;\rm abs} + \kappa_{\lambda;\rm sca})$

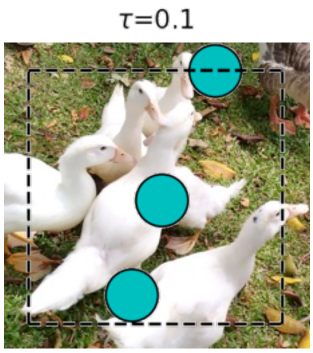
Mass density ($kg\ m^{-3}$)

Mass-weighted opacity ($m^2\ kg^{-1}$)

Radiative Transfer Equation (II)

$$\frac{dI_\lambda(s)}{\alpha_\lambda(s)ds} = -\frac{\alpha_\lambda(s)}{\alpha_\lambda(s)}I_\lambda(s) + \frac{j_\lambda(s)}{\alpha_\lambda(s)} \Rightarrow \boxed{\frac{d\boldsymbol{I_\lambda}(s)}{d\boldsymbol{\tau_\lambda}} = -\boldsymbol{I_\lambda}(s) + \boldsymbol{S_\lambda}(s)}$$

$$\boldsymbol{d\tau_\lambda} = \boldsymbol{\alpha_\lambda}(s)\boldsymbol{ds} \quad ; \quad \boldsymbol{\tau_\lambda} = \int \boldsymbol{\alpha_\lambda}(s)\boldsymbol{ds}$$



Optical thickness (no dimension)

$\tau_\lambda << 1$ … optically thin

Radiative Transfer Equation (II)

$$\frac{dI_\lambda(s)}{\alpha_\lambda(s)ds} = -\frac{\alpha_\lambda(s)}{\alpha_\lambda(s)}I_\lambda(s) + \frac{j_\lambda(s)}{\alpha_\lambda(s)} \Rightarrow \boxed{\frac{d\boldsymbol{I_\lambda}(s)}{d\boldsymbol{\tau_\lambda}} = -\boldsymbol{I_\lambda}(s) + \boldsymbol{S_\lambda}(s)}$$

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$$d\tau_\lambda = \alpha_\lambda(s)ds \quad ; \quad \tau_\lambda = \int \alpha_\lambda(s)ds$$

τ=3.0

Optical thickness (no dimension)

$\tau_\lambda << 1$ … optically thin

$\tau_\lambda >> 1$ … optically thick

Radiative Transfer Equation (II)

$$\frac{dI_\lambda(s)}{\alpha_\lambda(s)ds} = -\frac{\alpha_\lambda(s)}{\alpha_\lambda(s)}I_\lambda(s) + \frac{j_\lambda(s)}{\alpha_\lambda(s)} \rightarrow \boxed{\frac{dI_\lambda(s)}{d\tau_\lambda} = -I_\lambda(s) + S_\lambda(s)}$$

$$d\tau_\lambda = \alpha_\lambda(s)ds \quad ; \quad \tau_\lambda = \int \alpha_\lambda(s)ds$$

τ=0.1

Optical thickness (no dimension)

$\tau_\lambda << 1$ … optically thin

$\tau_\lambda >> 1$ … optically thick

(if the particles are very small)

Radiative Transfer Equation (II)

$$\frac{dI_\lambda(s)}{\alpha_\lambda(s)ds} = -\frac{\alpha_\lambda(s)}{\alpha_\lambda(s)}I_\lambda(s) + \frac{j_\lambda(s)}{\alpha_\lambda(s)} \rightarrow \boxed{\frac{dI_\lambda(s)}{d\tau_\lambda} = -I_\lambda(s) + S_\lambda(s)}$$

$$d\tau_\lambda = \alpha_\lambda(s)ds \quad ; \quad \tau_\lambda = \int \alpha_\lambda(s)ds$$

Optical thickness (no dimension)

$\tau_\lambda << 1$ … optically thin

$\tau_\lambda >> 1$ … optically thick

τ=3.0

(if the particles are very small)

Radiative Transfer Equation (II)

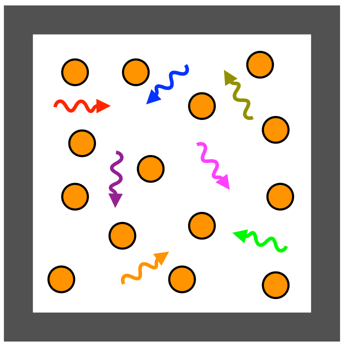
$$S_\lambda(s) = \frac{j_\lambda(s)}{\alpha_\lambda(s)}$$

$$\boxed{\frac{dI_\lambda(s)}{d\tau_\lambda} = -I_\lambda(s) + S_\lambda(s)}$$

Source Function (same dimension as I_λ)

$$S_\lambda = B_\lambda(T)$$

in
- **Radiation Equilibrium**
- **Local Thermal Equilibrium** at a single temperature (Kirchhoff's law)



Radiation Equilibrium

Radiative Transfer Equation (III)

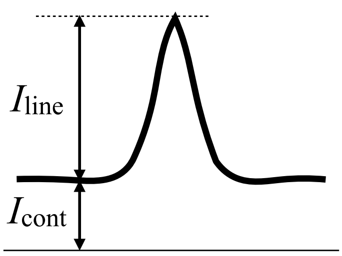
$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + B_\lambda(T)$$ (i.e., in a spatially uniform condition)

$$\boxed{\boldsymbol{I_{\lambda(\mathrm{out})} = e^{-\tau_\lambda} I_{\lambda(\mathrm{in})} + (1 - e^{-\tau_\lambda}) B_\lambda(T)}}$$

$$\tau_\lambda << 1 \longrightarrow I_{\lambda(\mathrm{out})} \sim (1 - \tau_\lambda) I_{\lambda(\mathrm{in})} + \tau_\lambda B_\lambda(T)$$

$$\tau_\lambda >> 1 \longrightarrow I_{\lambda(\mathrm{out})} \sim B_\lambda(T)$$

★ Applications (I)



$$I_{\mathrm{line+cont}} = (1 - e^{-(\tau_{\mathrm{line}} + \tau_{\mathrm{cont}})}) B_\lambda(T)$$

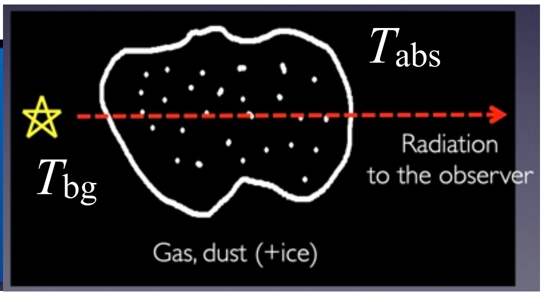
$$I_{\mathrm{cont}} = (1 - e^{-\tau_{\mathrm{cont}}}) B_\lambda(T)$$

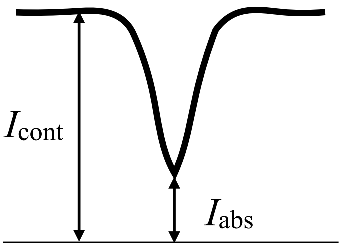
$$I_{\mathrm{line+cont}} > I_{\mathrm{cont}}$$

but not $\tau_{\mathrm{line+cont}}, \tau_{\mathrm{cont}} >> 1$ (otherwise, $I_{\mathrm{line+cont}} = I_{\mathrm{cont}} = B_\lambda(T)$)

★ Applications (II)

Absorption features are seen if ice/gas/dust:-
- lie between the observer & background source
- are *not* opaque *(optically thick)* at all wavelengths
- are colder than the background source



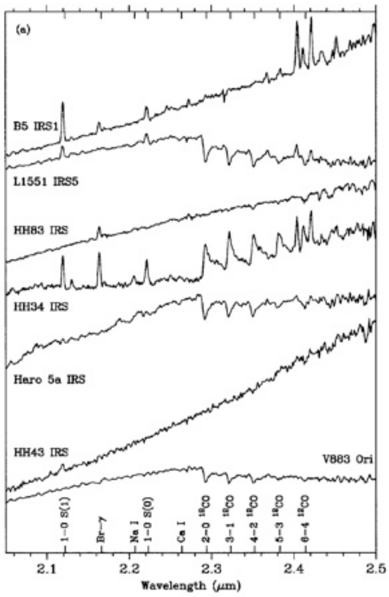


$$I_{\rm cont} = B_\lambda(T_{\rm bg})$$

$$I_{\rm abs} = e^{-\tau_{\rm abs}} B_\lambda(T_{\rm bg}) + (1 - e^{-\tau_{\rm abs}}) B_\lambda(T_{\rm abs})$$

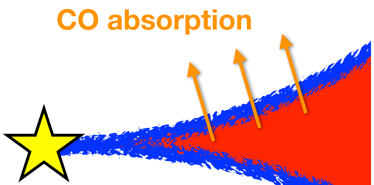
- $T_{\rm abs} < T_{\rm bg} \longrightarrow I_{\rm abs} < I_{\rm cont}$
- $T_{\rm abs} > T_{\rm bg} \longrightarrow I_{\rm abs} > I_{\rm cont}$

★ Applications (II)

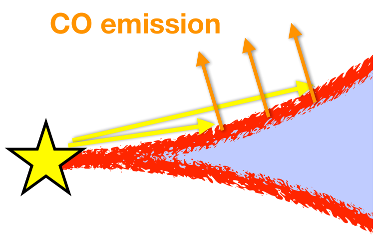


Reipurth+ (1997)

Internally heated disk



Externally heated disk

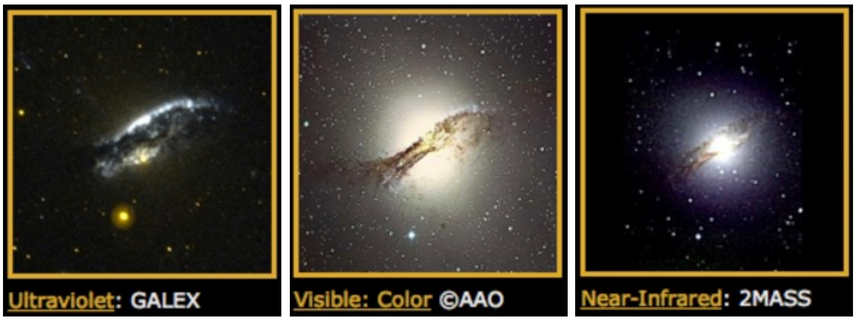


★ Applications (III): Measuring the amount of dust/gas

$$I_{\lambda(\text{out})} = e^{-\tau_\lambda} I_{\lambda(\text{in})} + (1 - e^{-\tau_\lambda}) B_\lambda(T)$$

Stars (T_{eff}=3000-50000K) — **Interstellar Dust** (T_{eff} ~ 30K)

$B_{0.55\mu m}(T)$ = 4×10^{11}-3×10^{15} — $<1\times10^{-250}$



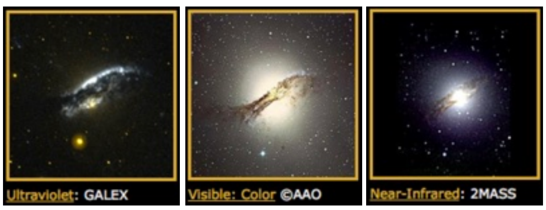
★ Applications (III): Measuring the amount of dust/gas

$$I_{\lambda(\text{out})} = e^{-\tau_\lambda} I_{\lambda(\text{in})} + (1 - e^{-\tau_\lambda}) B_\lambda(T)$$

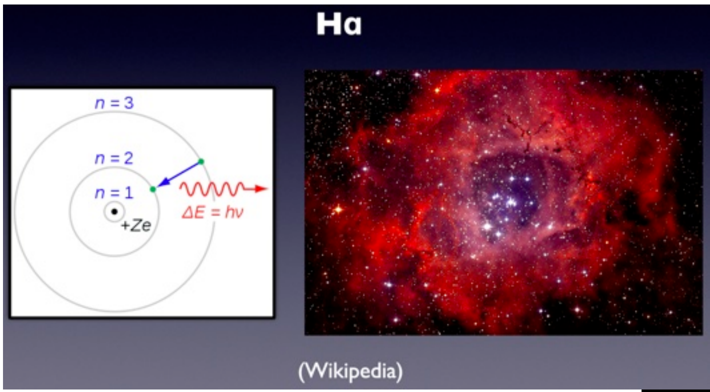
$$\tau_\lambda = -\log \frac{I_{\lambda(\text{out})}}{I_{\lambda(\text{in})}} = \int \alpha_\lambda(s)ds = \kappa_\lambda \int \rho(s)ds$$

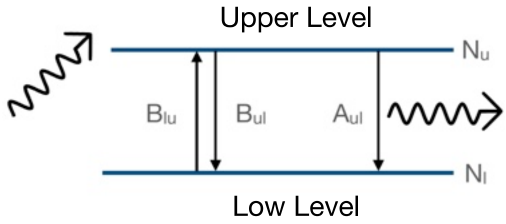
Mass Column Density (kg m^{-2})

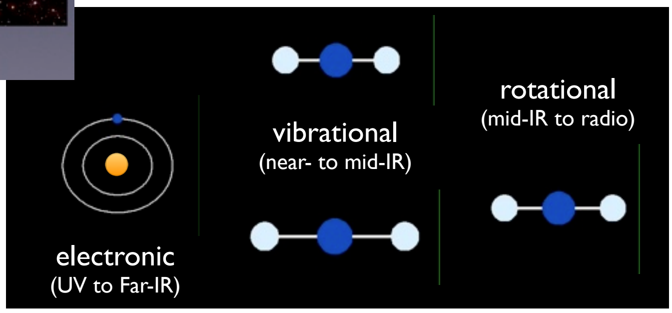
$$\int \rho(s)ds = -\frac{1}{\kappa_\lambda} \log \frac{I_{\lambda(\text{out})}}{I_{\lambda(\text{in})}}$$



Line Emission & Absorption

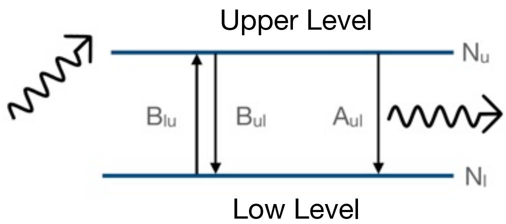






Line Emission & Absorption

$$\frac{dI_\nu(s)}{ds} = -\alpha_\nu(s) I_\nu(s) + j_\nu(s)$$



Photon energy

Number density

$$j_\nu = \frac{h\nu}{4\pi} n_{\rm u} A_{\rm ul} \phi(\nu)$$

Line profile function

Nomalization for solid angle

Einstein Coefficients

$$\alpha_\nu = \frac{h\nu}{4\pi} (n_{\rm l} B_{\rm lu} - n_{\rm u} B_{\rm ul}) \phi(\nu)$$

(cf. Einstein Relations)

$$A_{\rm ul} = \frac{2h\nu^3}{c^2} B_{\rm ul}$$

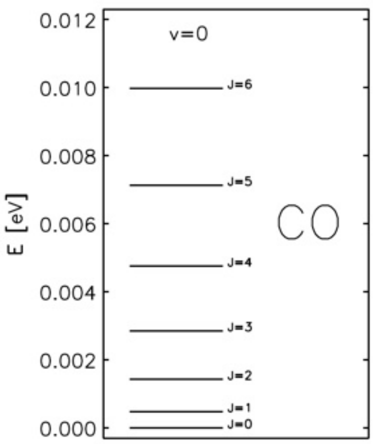
$$g_{\rm l} B_{\rm lu} = g_{\rm u} B_{\rm ul}$$

Level Populations

In the local thermal equilibrium (LTE),

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-\Delta E/k_B T} = \frac{g_u}{g_l} e^{-h\nu_0/k_B T}$$

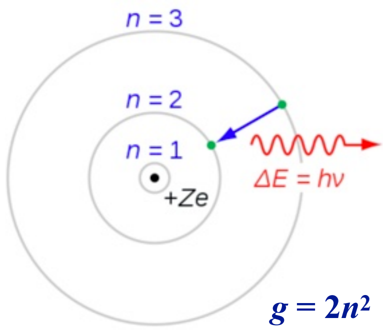
Statistical weight



$g = 2J+1$

- **Intensity = $B_\lambda(T)$ and $B_\nu(T)$**

  if optically thick



$g = 2n^2$

Level Populations

Thermal collisions with electrons or atomic/molecular hydrogen

$$C_{ul} = n\, k_{ul}(T)\ ,\ C_{lu} = n\, k_{lu}(T)$$

Number density   Collisional rate coefficients

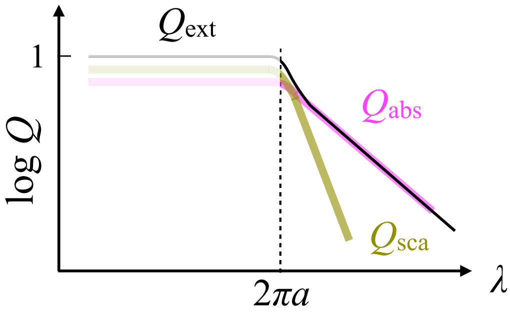
LTE: $C_{ul} = n\, k_{ul} >> A_{ul}$

Critical Density $n_{crit} = A_{ul} / k_{ul}$

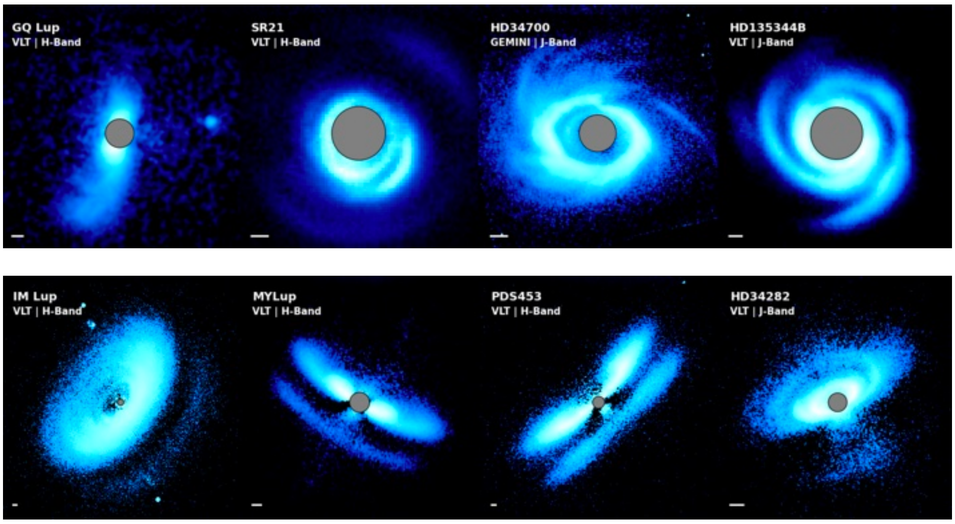
non-LTE: $C_{ul} = n\, k_{ul} << A_{ul}$

cf.) no time variation: $n_u C_{ul} = n_l C_{lu} \longrightarrow \frac{C_{lu}}{C_{ul}} = \frac{k_{lu}(T)}{k_{ul}(T)} = \frac{g_u}{g_l} e^{-h\nu_0/k_B T}$

Scattering

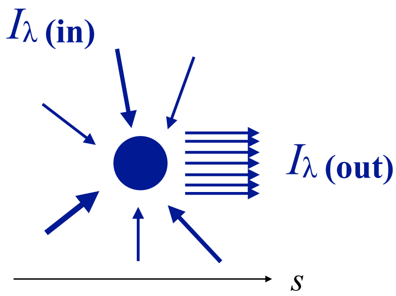


Scattering



Protoplanetary disks in scattered light (Benisty+ 2023)

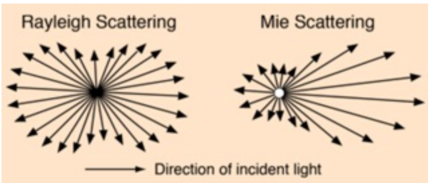
Scattering

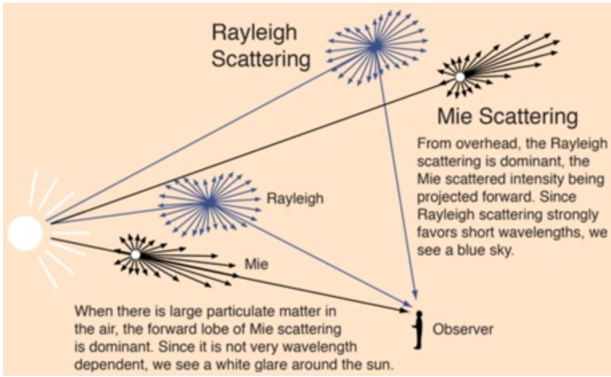


$$\frac{dI_\lambda(s)}{ds} = -\alpha_\lambda(s) I_\lambda(s) + j_\lambda(s) \; + \textbf{?}$$

This term is not easy to add as it depends on:

- I_λ from different directions
- Scattering phase function

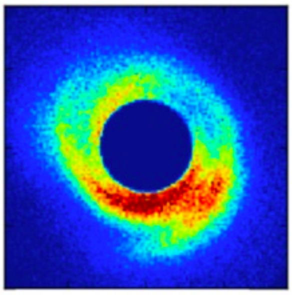




http://hyperphysics.phy-astr.gsu.edu/hbase/atmos/blusky.html

★ Monte-Carlo Simulations

- Using many photons
- Using random numbers to decide each direction or path length



(Takami+ 2014)