# Astrophysics Computation & GPU

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# Outlines

- (Ia) Fundamentals of collisional gas and collisionless particle dynamics in cosmological settings

  Collisionless particles: dark matter & stars

  Collisional ases: atomic particles

- DM provides gravity, Stars are visible and the tracers of structures, and atomic particles form stars, determining how many tracers out there
- (Ib) Wave-like Dark Matter when particles are extremely light and in Bose-Einstein condensation

- (II) Implementation of GPU Computation & Results to be given by Dr. Shive tomorrow morning

# Stat. Mech. of Particles

- Both collisional gases and collisionless particles obey the Boltzmann equation:

$$d_t f(\boldsymbol{x},\boldsymbol{v},t) \equiv \partial_t f + \boldsymbol{v}\cdot\partial_x f + \boldsymbol{a}\cdot \partial_v f = \nu_c\,[C_+(f)-C_-(f)],$$

where $f$ is the probability density in 6D phase space, and $C_+(f)-C_-(f)$ represents the birth and death processes

- Conservation Laws of Gases and Particles

$$\partial_t D + \nabla\cdot\boldsymbol{F} = 0,$$

*where D* can be a vector density, and ***F*** be a tensor flux

# Collisional Gases

- In a highly collisional limit, $\nu_c\,[C_+(f)-C_-(f)]$ dominates the Boltzmann equation to the leading order,

$$\partial_t f + \boldsymbol{v}\cdot\partial_{\boldsymbol{x}} f + \boldsymbol{a}\cdot\,\partial_{\boldsymbol{v}} f = \nu_c\,[C_+(f)-C_-(f)]$$

and particles are always in local thermal equilibrium, described by the Maxwell-Boltzmann distribution,

$$f\,(x,v,t) = n(x,t)\,\exp[-mv^2/2T(x,t)]$$

# Collisionless Particles

- In the collisionless limit, $\nu_c$ is negligible,

$$\partial_t f + \boldsymbol{v}\cdot\partial_{\boldsymbol{x}} f + \boldsymbol{a}\cdot \partial_{\boldsymbol{v}} f = \nu_c \left[C_+(f)-C_-(f)\right]$$

- The distribution function satisfies a 6D phase-space conservation law:

$$\partial_t f + \partial_x \cdot(\boldsymbol{v}\, f) + \partial_v \cdot(\boldsymbol{a}\, f) = 0$$

Here $\boldsymbol{a}$ is replaced by the mean force per mass **F**/m. We will come back to discuss the subtlety of **F**.

# Laws of Collisional Gas Dynamics

- Macroscopic physical laws are normally derived from conservation laws, such as energy-momentum conservation, charge conservation, etc., and/or from symmetry, such as space isotropy and Lorentz symmetry
- The gas equation of motion can be derived from both conservation laws and Lorentz symmetry,

$$\partial_\alpha T^{\alpha\beta} = 0,$$

where $T^{\alpha\beta}$ is the 4x4 energy-momentum tensor, or

$$\partial_t T^{0\beta} + \partial_i (T^{i\beta} c) = 0.$$

- Indices α and β contain the Lorentz symmetry

- Consider a local observer moving with a gas element of mesoscopic volume $(\Delta x)^3$. The observer finds the gas element has no velocity, and hence $T^{00'}$ is the thermodynamics internal energy density $e'$, and $T^{ij'}$ is the pressure $p$.
- There is no energy flux $T^{0i'}$ since the gas is at rest, and no off-diagonal momentum flux $T^{ji'}$ due to isotropic velocity distribution
- The 4x4 tensor therefore becomes diagonal, $[T^{00'}, T^{11'}, T^{22'}, T^{33'}] = [e', p, p, p]$.
- This is the form of matter in the rest frame that enters the Einstein field equation in cosmology.

- However, such a $T^{\alpha\beta'}$ is not physical, since different observers have different velocities.
- Conservation laws, $\partial_\alpha T^{\alpha\beta} = 0$, only hold in inertial frame, and we should use $T^{\alpha\beta}$ instead of $T^{\alpha\beta'}$. The two are related by Lorentz transformation as

$$T = L^{-1}\, T'\, L$$

or $$(T(x))^{\alpha\beta}=(L^{-1}(-v(x)))^{\alpha}{}_{\gamma}\,(T(x))^{\gamma\lambda}\,(L(-v(x)))_{\lambda}{}^{\beta}$$

- Difference between SR and GR is that due to Equiv. Prin., conservation laws can be written in "non-inertial" frame in GR when the force is gravity
- The result has a simple form: $T^{\alpha\beta} = (e'+p)\, U^{\alpha}U^{\beta}$, where $U^{\alpha}$ is the 4-velocity of the gas element.

- The gas equation of motion therefore satisfies

$$\partial_\alpha((e'+p)U^\alpha U^\beta) = 0,$$

  and we have conservation of energy when $\beta=0$, and conservation of momentum when $\beta=i$.

- This equation describes SR fluid dynamics, and we can take the non-relativistic limit, $e' = \rho c^2 + e$, to arrive at the continuity equation for mass density,

  $\partial_t \rho c^2 + \nabla\cdot(V\rho c^2) = 0,$ to leading order, and

  $\partial_t(\rho \mathbf{V}^2/2+e) + \nabla\cdot[\mathbf{V}(\rho \mathbf{V}^2/2+e+p)] = 0,$ to next order,

  and the Navior-Stokes equation for momentum,

$$\partial_t(\rho\mathbf{V}) + \nabla\cdot(\rho\,\mathbf{V}\,\mathbf{V} + p\,\mathbf{I}) = 0.$$

- If gases are subject to gravitational forces, we add a force density, $\rho\mathbf{F}_g$, to the right-hand side.

- If the force is due to self-gravity, one can use Poisson's equation, $\nabla^2\varphi = \rho$, to absorb the force into the conservation law:

$$\partial_t(\rho \mathbf{V}) + \nabla \cdot [\rho\, \mathbf{V}\, \mathbf{V} + (\nabla\varphi\nabla\varphi - (\nabla\varphi)^2\mathbf{I}/2)) + p\, \mathbf{I}] = 0.$$

It guarantees the center of mass remains fixed.

- Energy equation becomes:

$$\partial_t(\rho(V^2/2+e+\varphi/2) + \nabla \cdot [\mathbf{V}(\rho(\mathbf{V}^2/2+e+\varphi)+p) + (\partial_t\varphi\nabla\varphi - \varphi\nabla\partial_t\varphi)/2\ ] = 0.$$

This guarantees that overall energy $\int \rho(V^2/2+e+\varphi/2)\, dr^3$ is conserved ➔ Virial theorem

# Dynamical Equation for Collisionless Particles

- $$\partial_t f + \partial_x \cdot (\mathbf{v}\, f) + \partial_v \cdot ((\mathbf{F}_g/m)\, f) = 0,$$

where $\mathbf{F}_g$ is self gravity.

- This equation is solved in 6-dimensional space

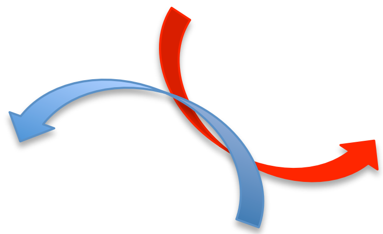
➔ Difficult if not impossible

- We normally adopt an equivalent method to solve this equation, by tracking the motion of N particles, i.e.,

$$m_i\, d_t^2\, \mathbf{x}_i = \Sigma_j\, \mathbf{F}_{ij}$$

- However, N-body approach is only approximate!

- The method is approximate because it can produce an undesirable effect, absent in a collisionless system ➔ large-angle scattering



- This trajectory yields two-body collision, and two particles can exchange energy and momentum
- The effect is absent in reality since particles are light, but in practice, the particle in the computation has a mass $10^8$ solar masses! For example, take $10^9$ particles in a $(100\ \mathrm{Mpc})^3$ volume ➔ there is only one particle per $(100\ \mathrm{kpc})^3$ volume! ➔ particle must be very massive
- Require a good method to avoid collision ➔ force regularization ➔ modified force law

# How to cope with an Expanding Universe?

- We rewrite the dynamical equation in a non-inertial frame, the co-moving frame: $\Delta \mathbf{x} = \mathbf{\Delta r}/a(t)$
- In this reference frame, total mass of the system is conserved, and matter appears at rest, at least initially, until gravity becomes so strong to pull the matter together.
- The co-moving frame is an decelerating (or accelerating) frame, it will therefore create a fictitious force $\boldsymbol{F}_{fic}^{(1)} + \boldsymbol{F}_{fic}^{(2)}$, much like the centrifugal force and Coriolis force in a rotating frame.
- When chosen properly, $\boldsymbol{F}_{fic}^{(1)}$ equals averaged $<\boldsymbol{F}_g>$, and particles see no force and can stay at rest.

# Equations for Collisional Gases & Collisionless Particles under Expansion

- Gas: change coordinate from (t, **r**) to (t, **x**) →

$$a^2\partial_t \rho + \nabla\cdot(\rho\,\mathbf{V}) = 0,$$

$$a^2\partial_t(\rho\mathbf{V}) + \nabla\cdot(\rho\,\mathbf{V}\,\mathbf{V} + p\,\mathbf{I}) = -\rho\nabla(\varphi_g + \varphi_p)$$

$$\nabla^2\varphi_g = 4\pi G a\,(\rho_g - \langle\rho_g\rangle)$$

- Particles:

fictitious force

$$a^{-1}d_t^2\mathbf{r}_i = d_t^2\mathbf{x}_i + 2H\,d_t\mathbf{x}_i + (d_t^2 a/a)\mathbf{x}_i = a^{-1}(\Sigma_j\,\mathbf{F}_{ij} + \mathbf{F}_g)/m_i$$

$$|\Phi_{ij}| = Gm_i m_j/a^2(x_{ij}^2 + \varepsilon^2)^{1/2},$$

where $H=(d_t a/a)$, and $\varepsilon$ is the soften length to avoid large angle scattering ➔ Particle-Particle method

# Practical Implementation for Particle Gravity

- $\mathbf{F}_{ij}$ calculates N(N-1)/2 pairs ➔ very time consuming
- The evaluation of $\mathbf{F}_{ij}$ can be efficiently performed by solving Poisson equation

$$\nabla^2 \varphi(x_i) = 4\pi G\, a\, (\Sigma_j\, m_j \delta(x_i - x_j) - \rho_0)$$

- But this force is singular at $x_i = x_j$, and so one must replace $\delta(x_i - x_j)$ by a smooth Gaussian $G(x_i - x_j)$ so as to impose force regularization ➔ Particle-Mesh method
- When many particles in one cell ➔ PPPM method

# BEC Dark Matter with Extremely Light Particles

- The average mass density in universe is well constrained, $\rho_0 = m\, n_0 = 10^9$ eV/m$^3$, and when $m \rightarrow 0$, then $n_0 \rightarrow \infty$
- If particles are bosons, they tend to be at the ground state, and probability for them to stay in the ground state depends on a number-density-dependent chemical potential, or critical temperature.
- When $T << T_{crit}(n) \sim n^{2/3}$, the probability is 100%
- When $m \sim 10^{-22}$ eV, one finds $n_0 \sim 10^{25}$/c.c., 300 times denser than water, and $T_{crit}(n_0) \sim$ keV
- Particles this light must be in BEC coherent state

# Dark Matter Wave

- Coherence means all bosons share the same wave function, but the wave function may exhibit very little observable wave features if particle mass is not sufficiently small, because the de Broglie wavelength is too small
- One example of such a BEC DM candidate is axion, having particle mass ranging from $10^{-6}$ eV to $10^{-10}$ eV, and they behave like particles on galaxy scales
- However, when $m \sim 10^{-22}$ eV, the de Broglie wavelength can range from 1 kpc with $\sigma \sim 20$ km/s in dwarf galaxies, to 10 pc with $\sigma \sim 2000$ km/s in galaxy cluster

- Equation of motion of the coherent wave function is simply described by the Schrodinger equation:

  $[i\hbar a^2 \partial_t - \hbar^2 \nabla^2/2m + \varphi]\,\psi = 0,$

  and the potential is due to self-gravity

  $\nabla^2 \varphi = 4\pi G a m[|\psi|^2 - <|\psi|^2>]$

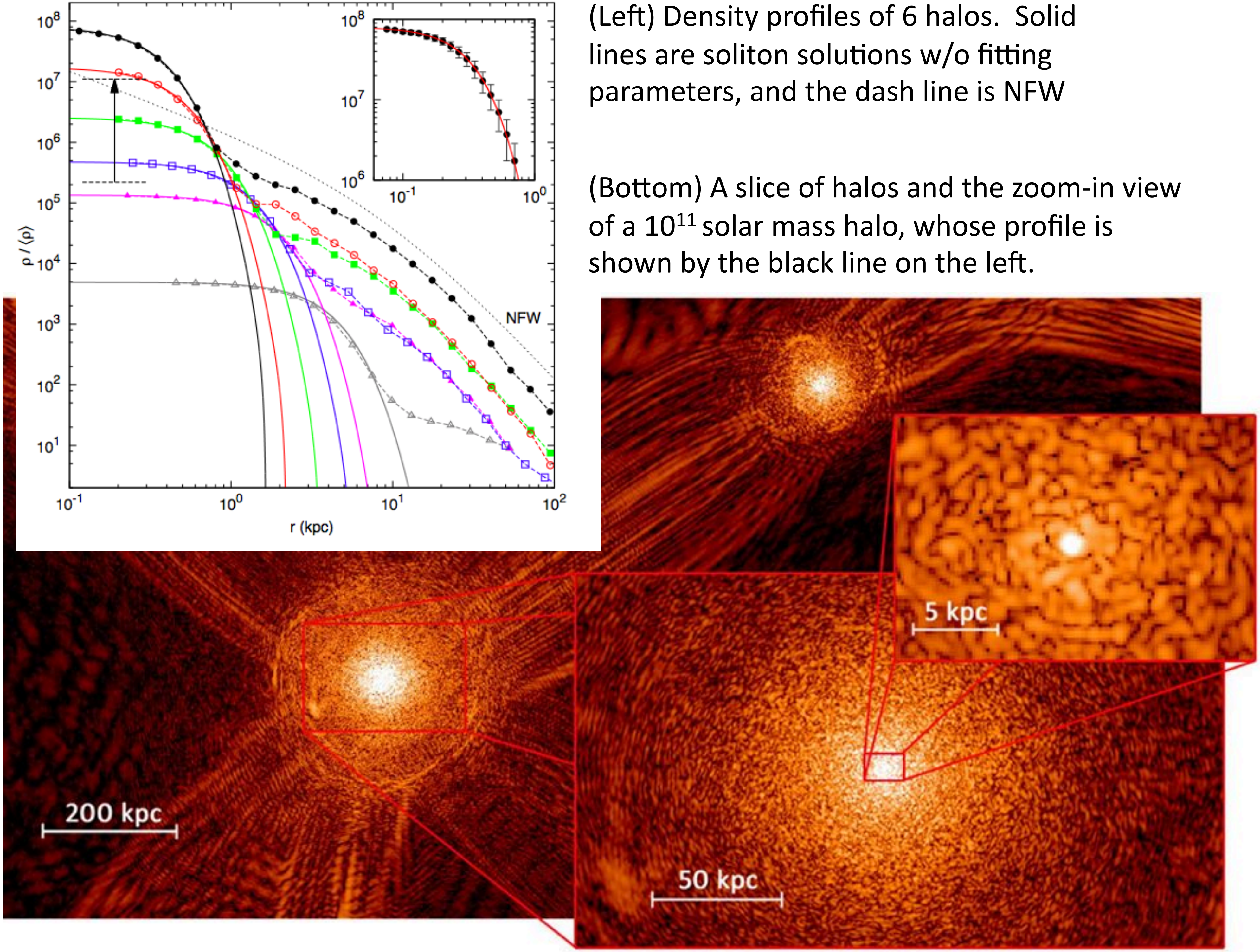
- The equations look simple, but difficult to evolve in time, because it is a diffusion-like equation
- Generally, $\omega \sim k^2$, when one zooms in twice to resolve higher k, the required time step will be 4 times smaller to resolve time dependence

# Features of ψDM Structures

- These equations admit a very stable nonlinear ground-state solution, a soliton, appearing like the ground state of a simple harmonic oscillator
- These equations are scale free, like CDM, the solution is self-similar: $\Delta x \Rightarrow b^{-1}\Delta x$, $\rho \Rightarrow b^4\rho$
- Because of the uncertainty principle, the solution has no density singularity
- Due to the background density, perturbations have a Jeans length $\lambda_{j,comove} \sim (a)^{-1/4}\, m^{-1/2}$, producing a linear power spectrum like WDM with high-k cutoff
- Halos have full-modulation small-scale density granules, but on average obey NFW profile ➔ Potential impact on strong lensing

(Left) Density profiles of 6 halos. Solid lines are soliton solutions w/o fitting parameters, and the dash line is NFW

(Bottom) A slice of halos and the zoom-in view of a $10^{11}$ solar mass halo, whose profile is shown by the black line on the left.



# Summary

- Collisional gases and collisionless particles both obey conservation laws, which is the foundation of almost all physical laws
- Gas dynamical equation can be derived from conservation laws and Lorentz symmetry
- Co-moving coordinate is a convenient non-inertial coordinate used for computation
- Particle dynamics requires force regularization to ensure collisionless since the particle mass is astronomically huge in computation
- Some features of ΨDM
- We will discretize the PDEs to obtain time-dependent solutions in the next lecture