From Galactic Halos to the Reimann Zeta Function: an Adventure in Applied Math

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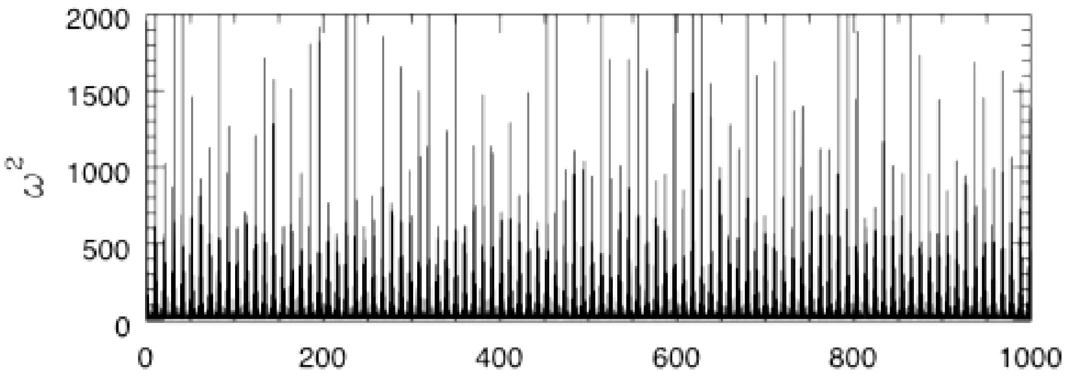


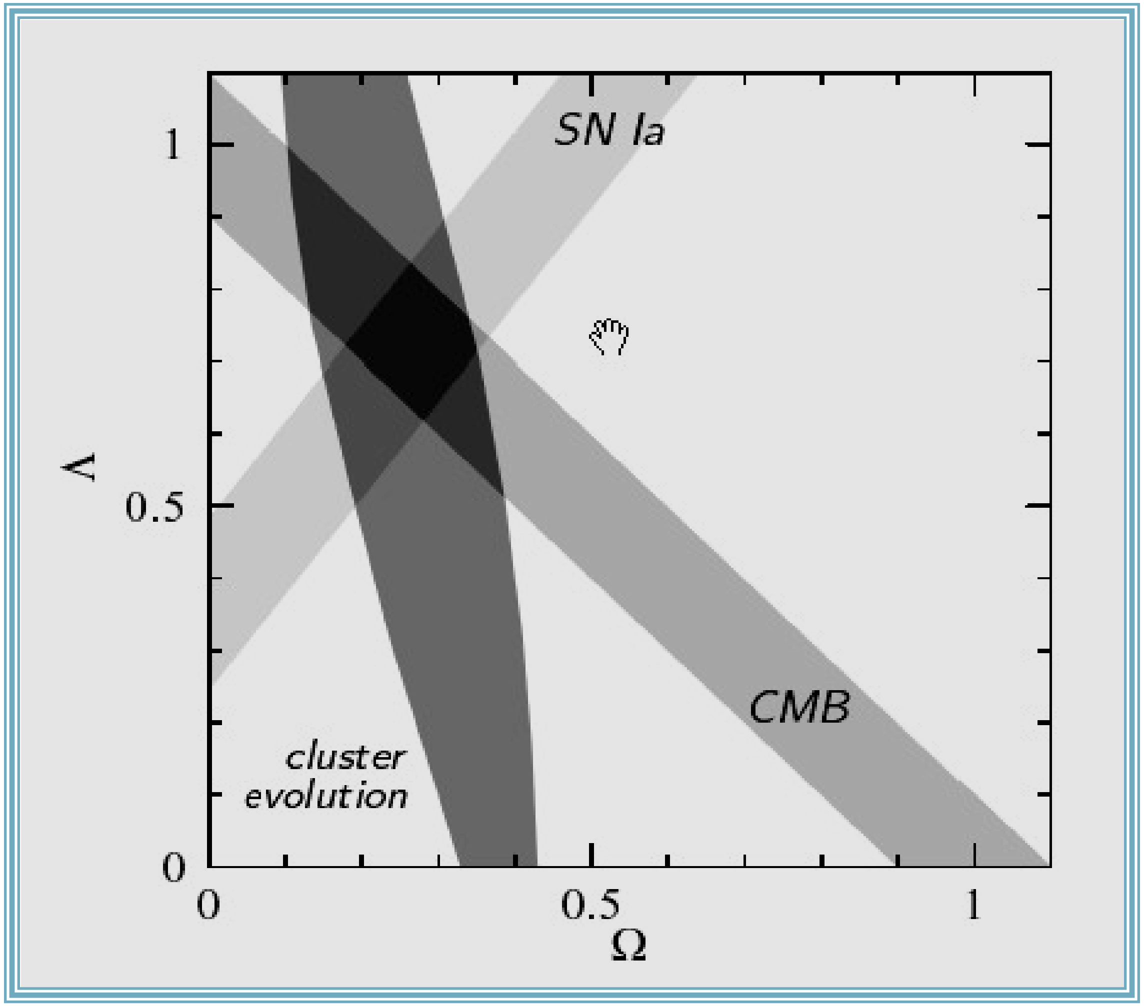
Homage to CC Lin

and with warm regards for FH Shu…

OUTLINE

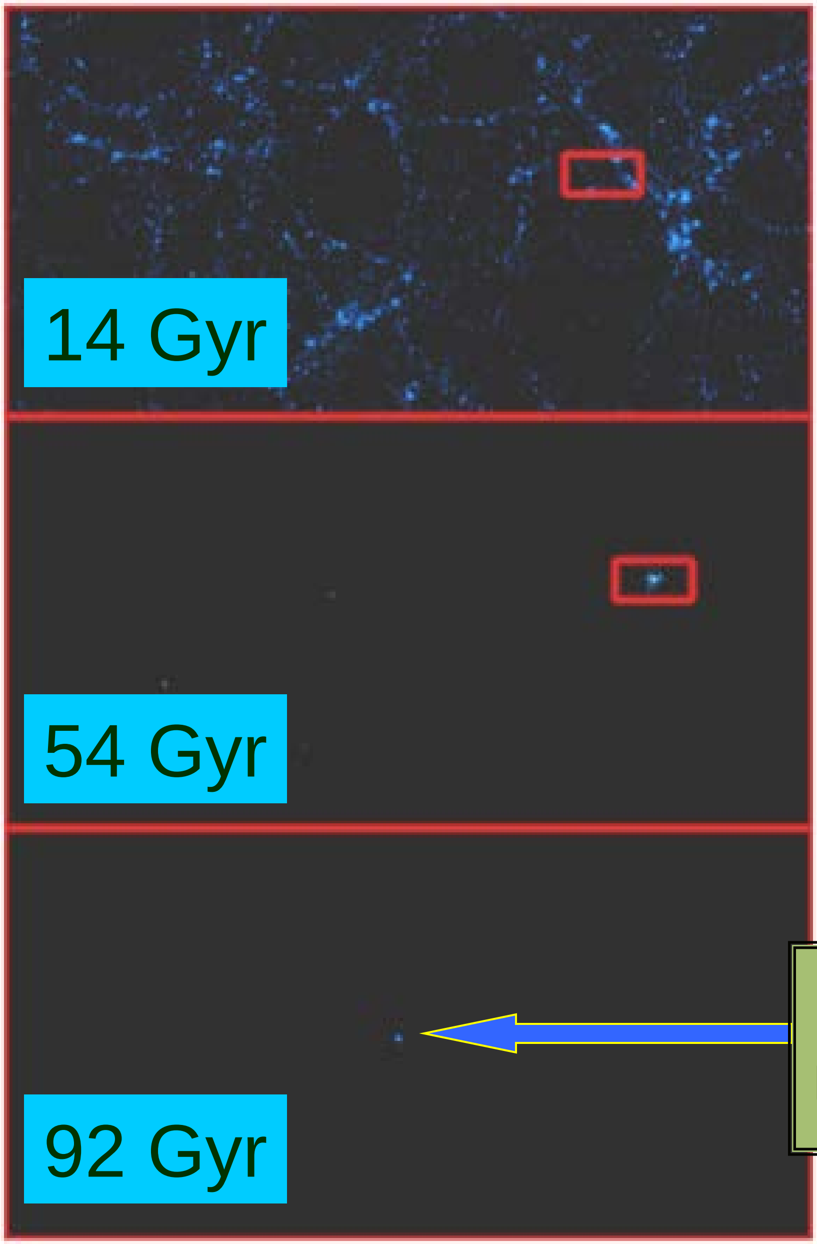
- Astrophysical motivation
- Orbits in spherical galactic halos, bulges
- Triaxial potentials and triaxial orbits
- Instabilities in triaxial systems
- Hill’s equation and its generalization to include stochastic variations
- Infinite strings of random matrices
- Astrophysical implications

What is the total mass of a galaxy? Why do dark matter halos have a nearly universal form?



Cosmological Parameters

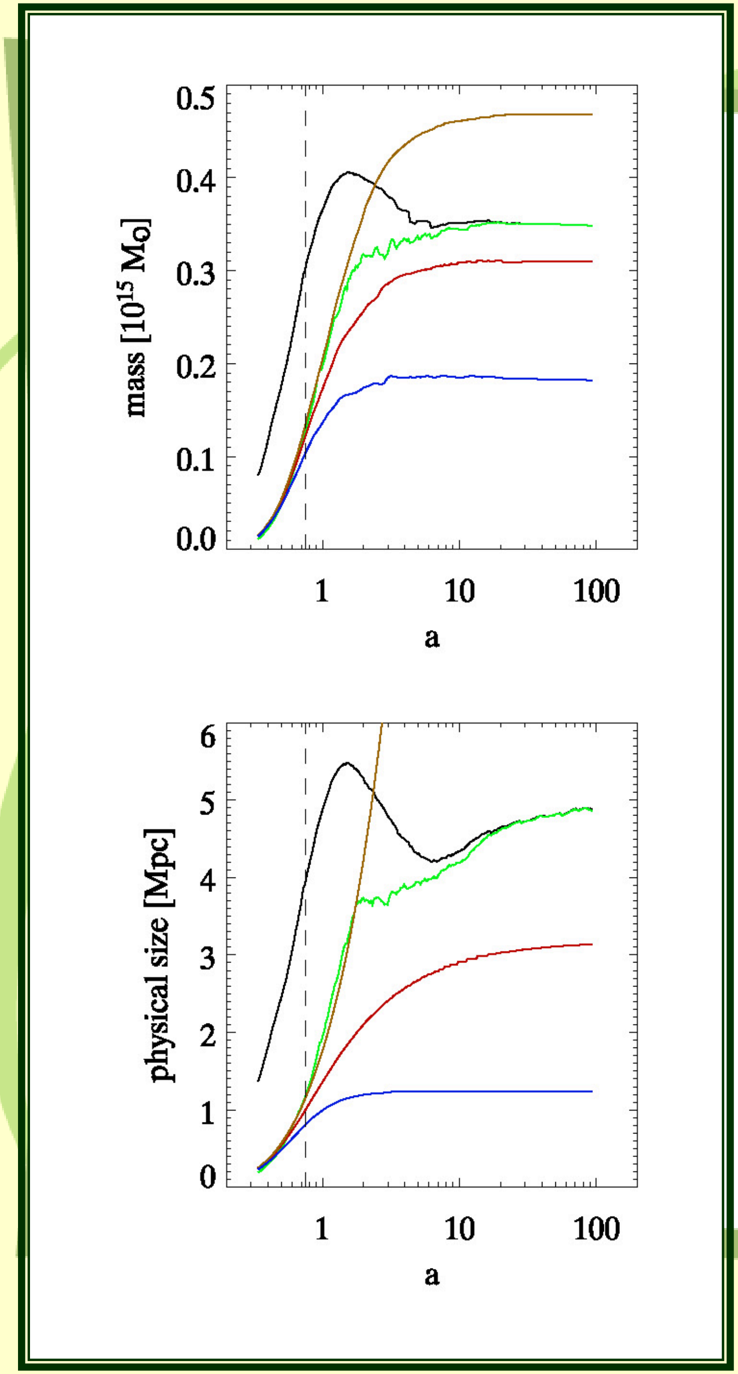

a = 0.244

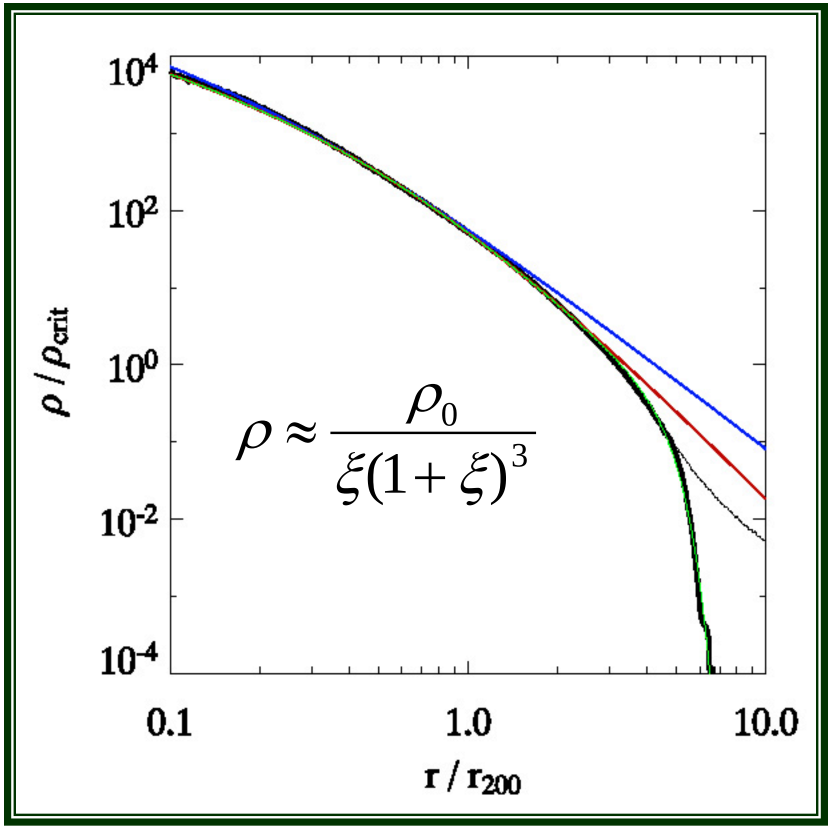


(M. Busha, F. Adams, et al. 2003, 2005, 2007)

Island Universe

Dark matter halos approach a well-defined asymptotic form with unambiguous total mass, outer radius, & density profile

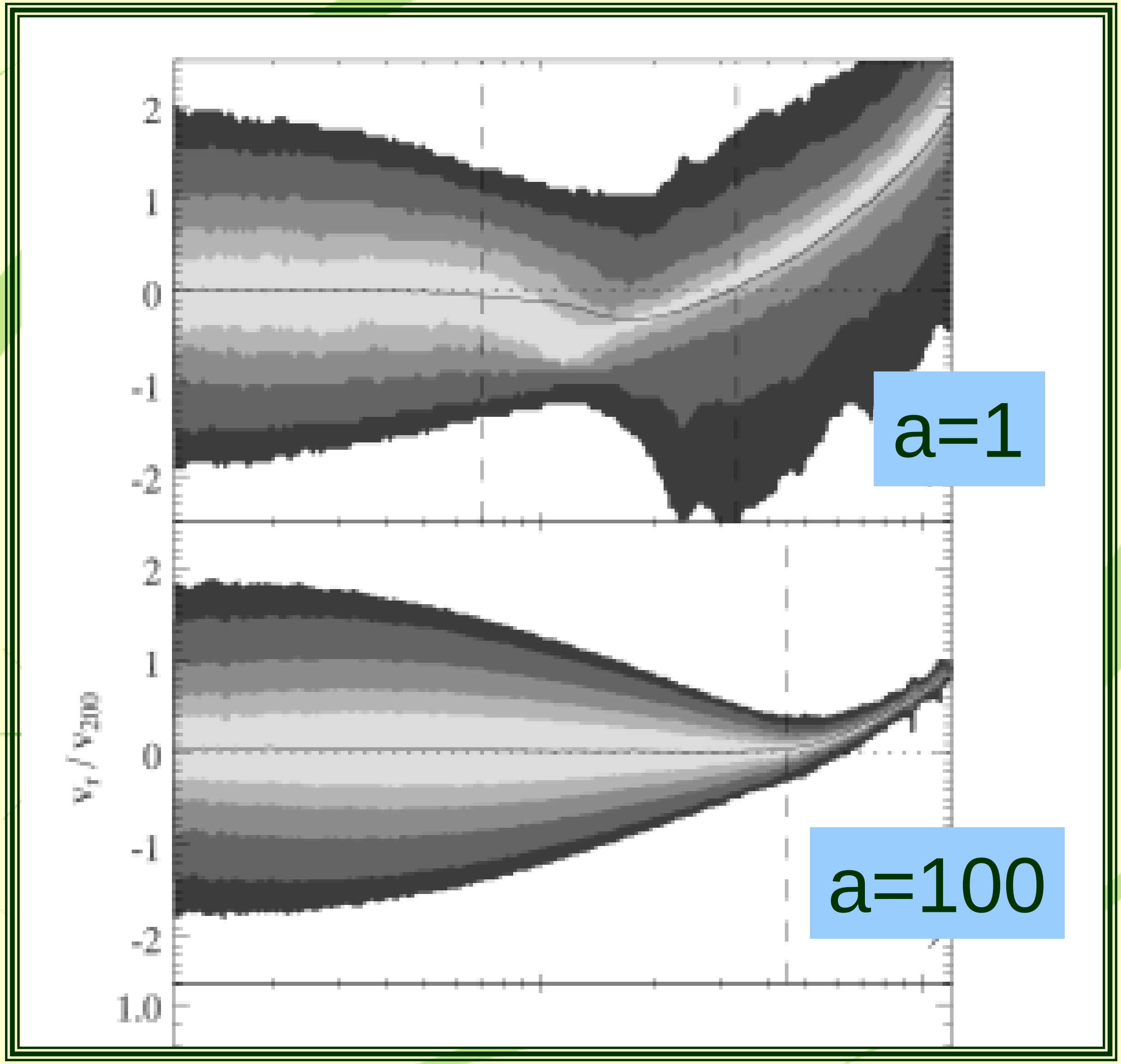




$$\rho \approx \frac{\rho_0}{\xi(1+\xi)^3}$$

(Busha et al. 2005)

Phase Space of Dark Matter Halo

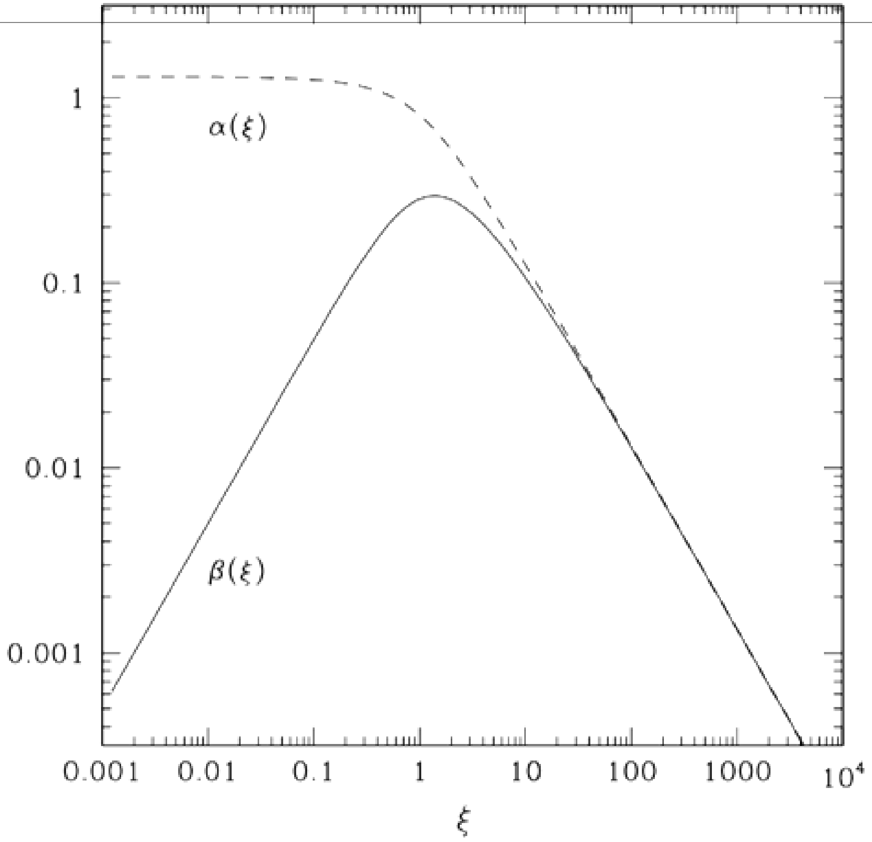


a=1

a=100

(M. Busha et al. 2005)

Spacetime Metric Attains Universal Form

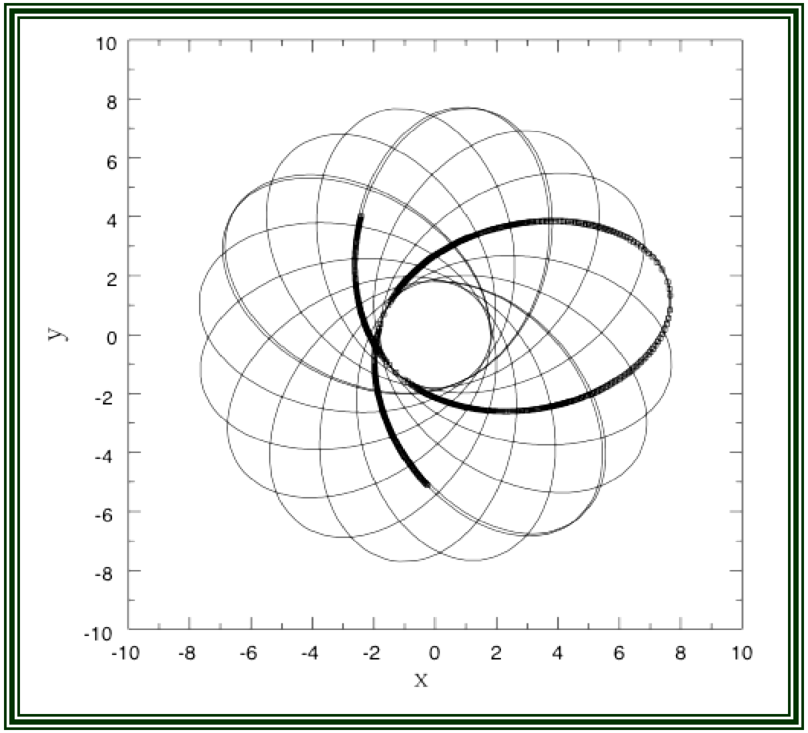


$$ds^2 = -[1 - A(r) - \chi^2 r^2]dt^2 + \frac{dr^2}{[1 - B(r) - \chi^2 r^2]} + r^2 d\Omega^2$$

WHY THESE ORBITS?

- Most of the mass is in dark matter
- Most dark matter resides in these halos
- Halos have the universal form found here (nfw/hq) for most of their lives
- Most orbital motion that will EVER occur will be THIS orbital motion $\boxed{\left(\textit{factor of } 10^{74} \right)}$

Spherical Limit:
Orbits look like Spirographs



Orbits in Spherical Potential

$$\rho = \frac{\rho_0}{\xi(1+\xi)^3} \Rightarrow \Psi = \frac{\Psi_0}{1+\xi}$$

$$\varepsilon \equiv \left|E\right| / \Psi_0 \quad \mathit{and} \quad q \equiv j^2 / 2\Psi_0 r_s^2$$

$$\varepsilon = \frac{\xi_1 + \xi_2 + \xi_1\xi_2}{(\xi_1 + \xi_2)(1 + \xi_1 + \xi_2 + \xi_1\xi_2)}$$

$$q = \frac{(\xi_1\xi_2)^2}{(\xi_1 + \xi_2)(1 + \xi_1 + \xi_2 + \xi_1\xi_2)}$$

$$q_{\max} = \frac{1}{8\varepsilon}\frac{\left(1+\sqrt{1+8\varepsilon}-4\varepsilon\right)^3}{\left(1+\sqrt{1+8\varepsilon}\right)^2}$$

(angular momentum of the circular orbit)

$$\xi_* = \frac{1-4\varepsilon+\sqrt{1+8\varepsilon}}{4\varepsilon}$$

(effective semi-major axis)

$$\frac{\Delta\theta}{\pi} = \frac{1}{2} + \left[(1+8\varepsilon)^{-1/4} - \frac{1}{2}\right]\left[1+\frac{\log(q/q_{\max})}{6\log 10}\right]^{3.6}$$

$$\lim_{q\to q_{\max}} \Delta\theta = \pi(1+8\varepsilon)^{-1/4}$$

(circular orbits do not close)

Limiting Forms

$$\varepsilon \to 0 \quad kepler \quad regime:$$

$$q_{\max} = \frac{1}{4\varepsilon} \quad and \quad \xi_* = \frac{1}{2\varepsilon} \quad and \quad \Delta\theta = \pi$$

$$\varepsilon \to 1 \quad central \quad regime:$$

$$q_{\max} = \frac{4}{27}(1-\varepsilon)^3 \quad and \quad \xi_* = \frac{2}{9}(1-\varepsilon)^2$$

These results determine the radiation exposure of a star, averaged over its orbit, as a function of energy and angular momentum:

$$\left\langle F_{fuv} \right\rangle \approx \frac{L_{fuv}}{8 r_s^2 \sqrt{q}} \frac{A \varepsilon^{3/2}}{\cos^{-1}\sqrt{\varepsilon} + \sqrt{\varepsilon}\sqrt{1-\varepsilon}}$$

$$where \quad 1 \leq A(q) \leq \sqrt{2}$$

Triaxial Density Distributions

- Relevant density profiles include NFW and Hernquist

$$\rho_{nfw} = \frac{1}{m(1+m)^2} \qquad \rho_{Hern} = \frac{1}{m(1+m)^3}$$

- Isodensity surfaces in triaxial geometry

$$m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \qquad a > b > c > 0$$

- In the inner limit both profiles scale as 1/r

$$m << 1 \quad \Rightarrow \quad \rho \propto \frac{1}{m}$$

Triaxial Potential

$$\Phi = \int_0^\infty du \frac{\psi(m)}{\sqrt{(u+a^2)(u+b^2)(u+c^2)}} \qquad \psi(m) = \int_\infty^{m^2} \rho(m)\, dm^2$$

- In the inner limit the above integral can be simplified to

$$\Phi = -I_1 + I_2$$

where I_1 is the depth of the potential well and the effective potential is given by

$$\boxed{I_2 = 2\int_0^\infty du \frac{\sqrt{\xi^2 u^2 + \Lambda u + \Gamma}}{(u+a^2)(u+b^2)(u+c^2)}}$$

ξ, Λ, Γ are polynomial functions of x, y, z, a, b, c

Triaxial Forces

$$F_x = \frac{-2\,\mathrm{sgn}(x)}{\sqrt{\left(a^2-b^2\right)\left(a^2-c^2\right)}}\ln\left(\frac{2G(a)\sqrt{\Gamma}+2\Gamma-a^2\Lambda}{2a^2\xi G(a)+\Lambda a^2-2a^4\xi^2}\right)$$

$$F_y = \frac{-2\,\mathrm{sgn}(y)}{\sqrt{\left(a^2-b^2\right)\left(b^2-c^2\right)}}\left[\sin^{-1}\left(\frac{\Lambda-2b^2\xi^2}{\sqrt{\Lambda^2-4\Gamma\xi^2}}\right)-\sin^{-1}\left(\frac{2\Gamma/b^2-\Lambda}{\sqrt{\Lambda^2-4\xi^2\Gamma}}\right)\right]$$

$$F_z = \frac{-2\,\mathrm{sgn}(z)}{\sqrt{\left(a^2-c^2\right)\left(b^2-c^2\right)}}\ln\left(\frac{2G(c)\sqrt{\Gamma}+2\Gamma-c^2\Lambda}{2c^2\xi G(c)+\Lambda c^2-2c^4\xi^2}\right)$$

$$G(u)=\xi^2u^4-\Lambda u^2+\Gamma$$

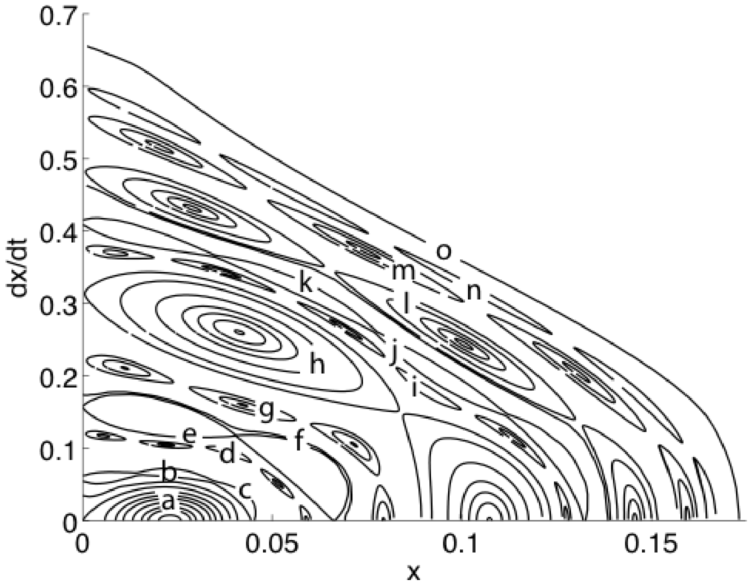
$$\xi^2=x^2+y^2+z^2$$

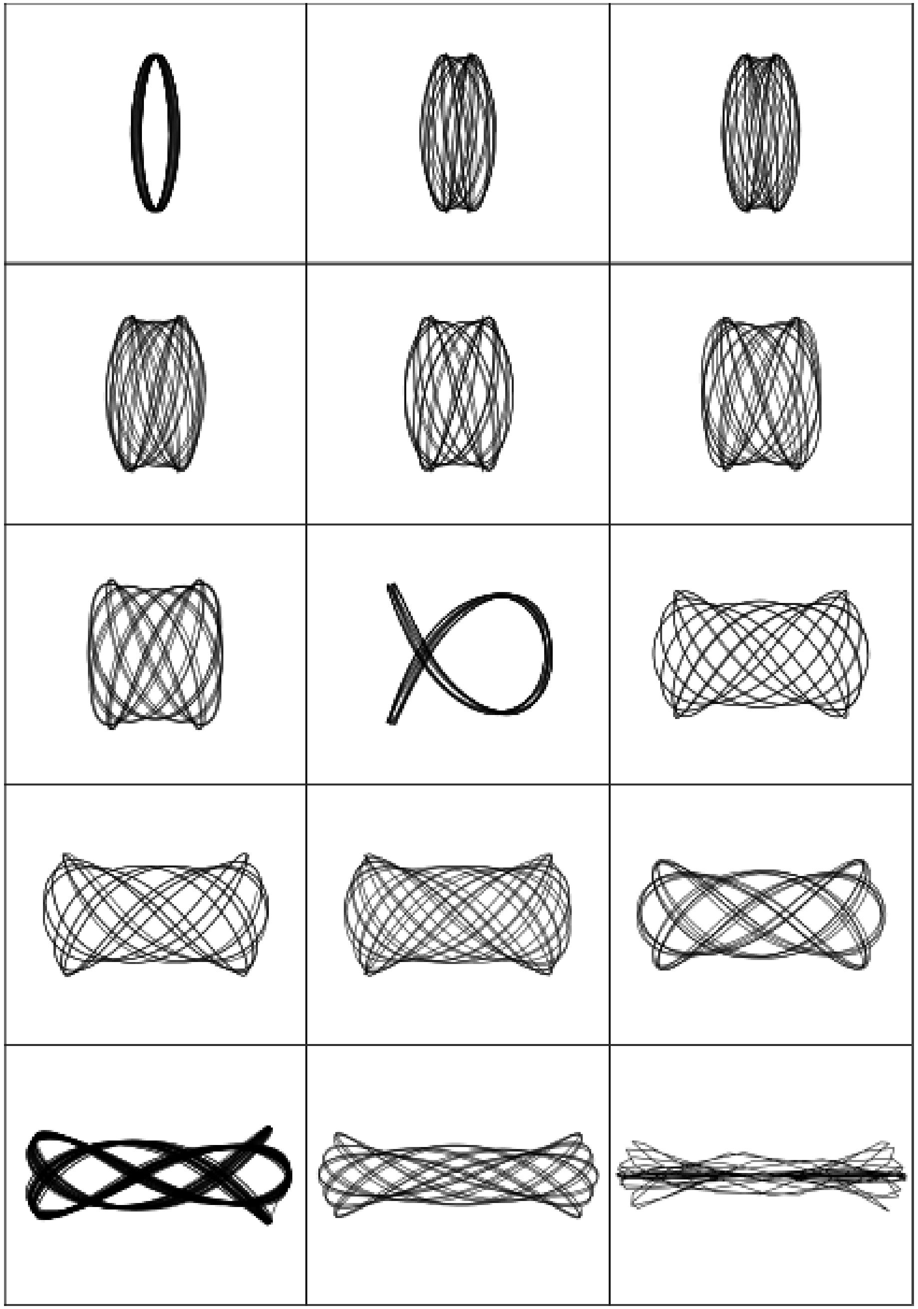
$$\Lambda=\left(b^2+c^2\right)x^2+\left(a^2+c^2\right)y^2+\left(a^2+b^2\right)z^2$$

$$\Gamma=b^2c^2x^2+a^2c^2y^2+a^2b^2z^2$$

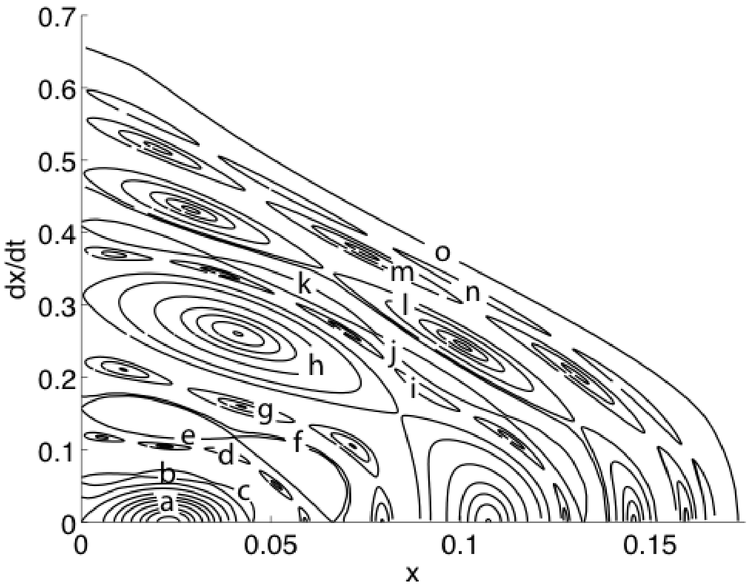
(Adams, Bloch, Butler, Druce, Ketchum 2007)

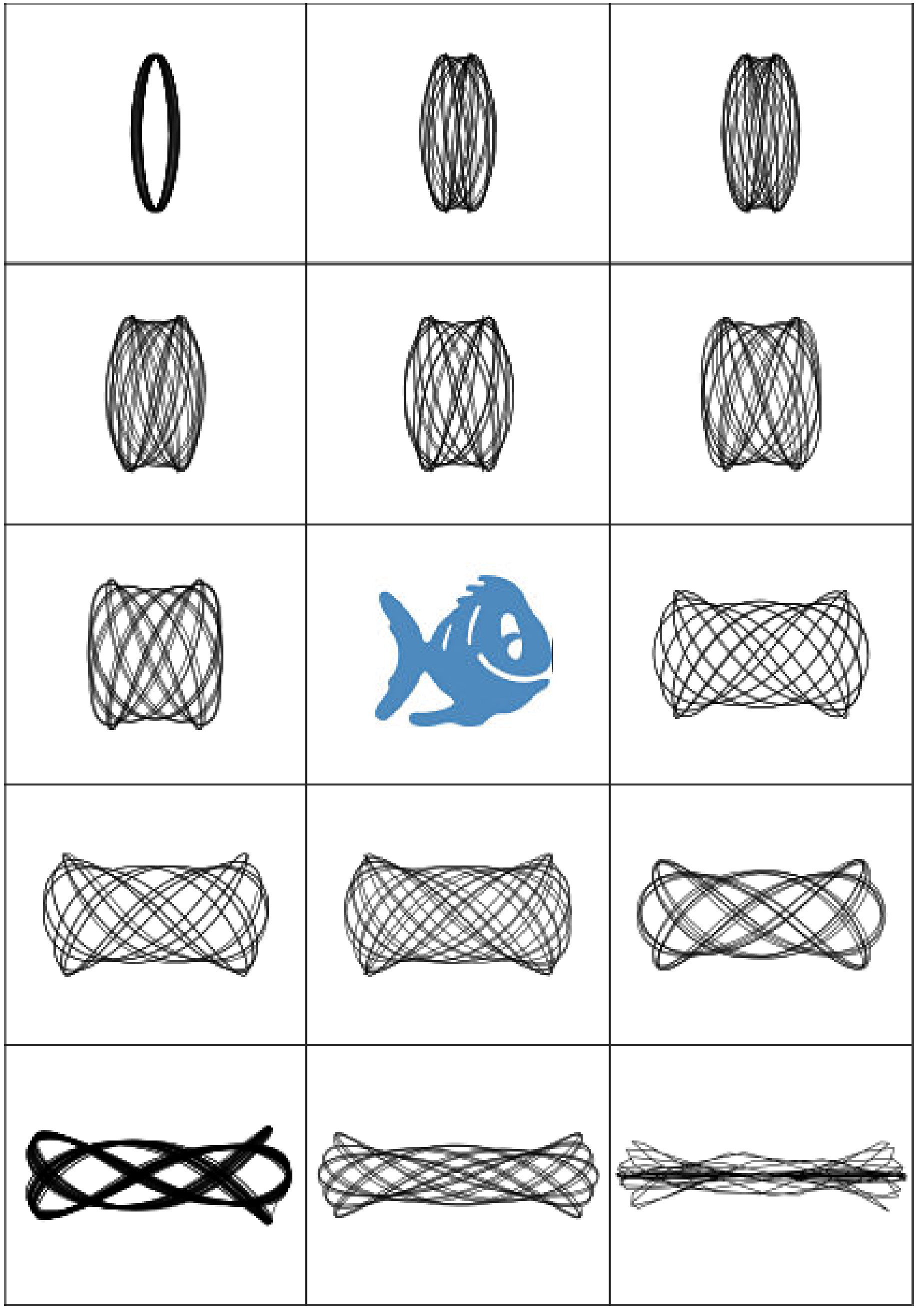
Orbit Gallery





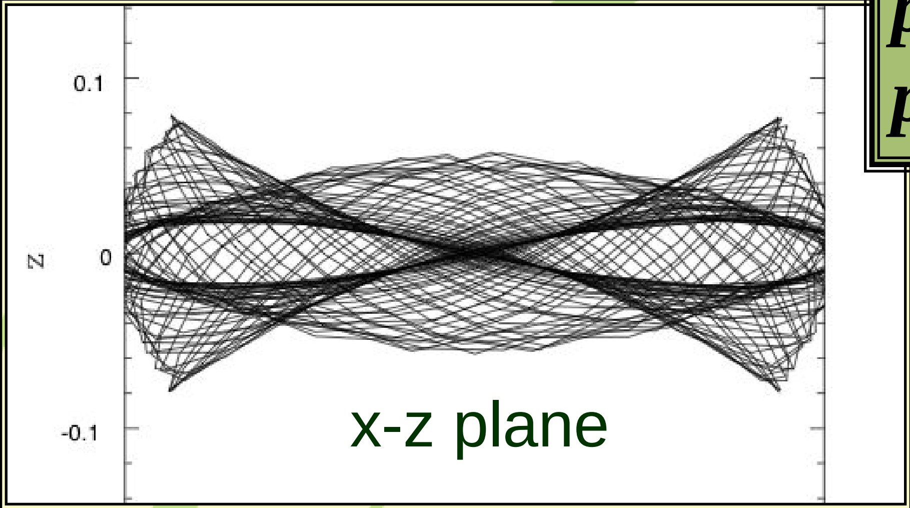
Orbit Gallery





INSTABILITIES

Orbits in any of the principal planes are unstable to motion perpendicular to the plane.



x-z plane

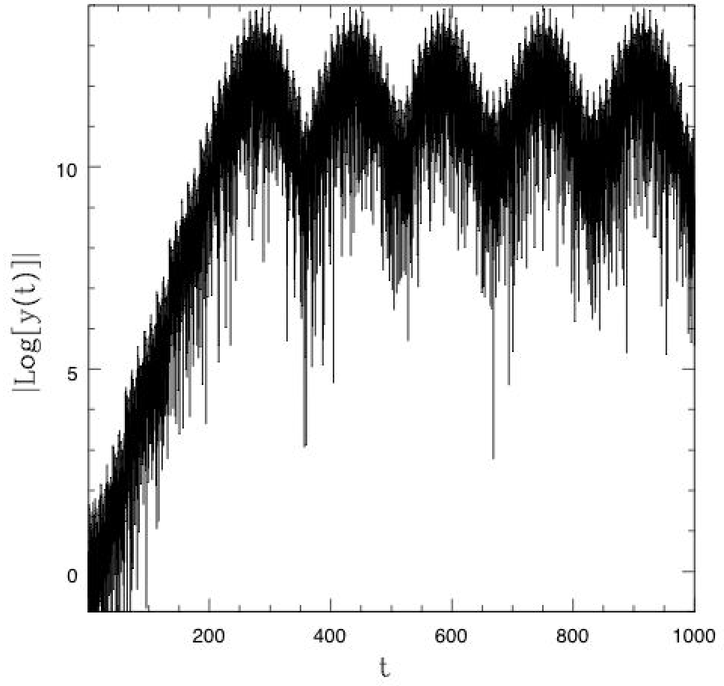
Unstable motion shows:

(1) exponential growth,

(2) quasi-periodicity,

(3) chaotic variations, &

(4) eventual saturation.



Perpendicular Perturbations

- Force equations in limit of small x, y, or z become

$$F_x \approx -\left(\frac{4}{a\left(\sqrt{c^2y^2+b^2z^2}+a\sqrt{y^2+z^2}\right)}\right)x$$

$$F_y \approx -\left(\frac{4}{b\left(\sqrt{c^2x^2+a^2z^2}+b\sqrt{x^2+z^2}\right)}\right)y$$

$$F_z \approx -\left(\frac{4}{c\left(\sqrt{b^2x^2+a^2y^2}+c\sqrt{x^2+y^2}\right)}\right)z$$

$$F_x \approx -\omega_x^2 x$$

$$F_y \approx -\omega_y^2 y$$

$$F_z \approx -\omega_z^2 z$$

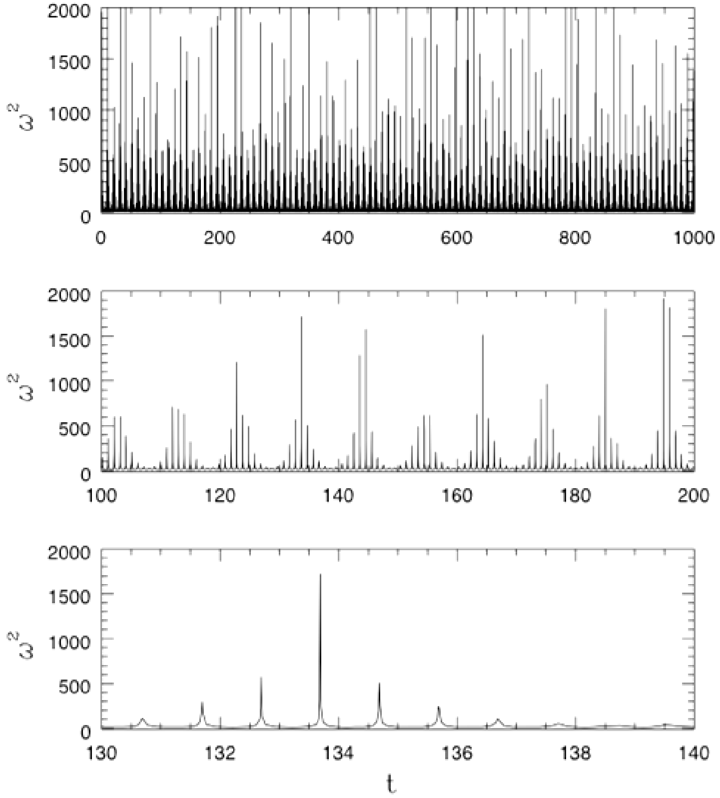
- Equations of motion perpendicular to plane have the form of Hill’s equation
- Displacements perpendicular to the plane are unstable

Hill’s equation

$$\frac{d^2y}{dt^2} + \frac{4/b}{\sqrt{c^2x^2 + a^2z^2} + b\sqrt{y^2 + z^2}}\,y = 0$$

$$\frac{d^2y}{dt^2} + \left[\lambda_k + q_k Q(\mu_k t)\right] y = 0$$

$$\frac{d^2y}{dt^2} + \omega^2(t)\,y = 0$$



Floquet's Theorem

For standard Hill's equations (including Mathieu equation) the condition for instability is given by Floquet's Theorem (e.g., Arfken & Weber 2005; Abramowitz & Stegun 1970):

$$|\Delta| \geq 2 \quad required \quad for \quad instability$$

$$where \quad \Delta \equiv y_1(\pi) + dy_2/dt(\pi)$$

$$\boxed{\frac{d^2y}{dt^2} + \left[\lambda + q\, Q(t)\right]y = 0}$$

Goal: Find analogous condition for stochastic case.

Principal Solutions

$$y_1(t=0)=1 \quad and \quad \frac{dy_1}{dt}(t=0)=0$$

$$y_2(t=0)=0 \quad and \quad \frac{dy_1}{dt}(t=0)=1$$

CONSTRUCTION OF DISCRETE MAP

$$\boxed{In\ general:\ \ y_k(t) = \alpha_k y_1(t) + \beta_k y_2(t)}$$

To match solutions from cycle to cycle, the coefficients are mapped via the 2x2 matrix:

$$\begin{bmatrix} \alpha_b \\ \beta_b \end{bmatrix} = \begin{bmatrix} h & (h^2-1)/g \\ g & h \end{bmatrix} \begin{bmatrix} \alpha_a \\ \beta_a \end{bmatrix}$$

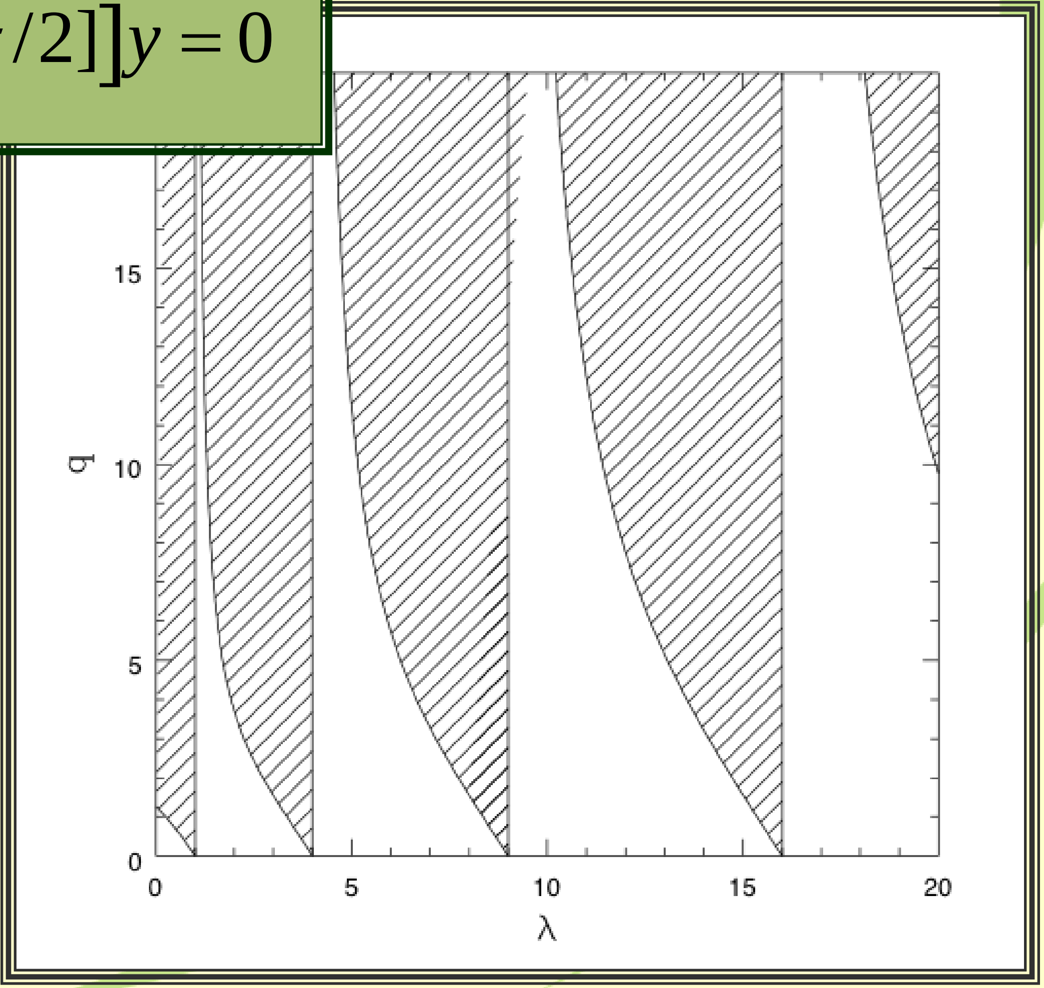
$$where\ h = y_1(\pi) \quad and \quad g = dy_1/dt(\pi)$$

Development of the solutions is determined by the eigenvalues of the 2x2 matrix…

Instability Strips for Hill’s Equation in Delta Function Limit

$$\frac{d^2y}{dt^2}+[\lambda+q\delta[t-\pi/2]]y=0$$

q given by distance of closest approach,
λ by outer turning pt



Stochastic Generalization of Hill's Equation

$$\frac{d^2y}{dt^2} + \left[\lambda_k + q_k Q(\mu_k t)\right] y = 0$$

$$(\lambda_k, q_k, \mu_k) \quad different \quad each \quad cycle$$

(assume random parameters are i.i.d.)

DISCRETE MAP for STOCHASTIC CASE

To match solutions from cycle to cycle, the coefficients are mapped using a set of 2x2 matrices:

$$\begin{bmatrix} \alpha_b \\ \beta_b \end{bmatrix} = \begin{bmatrix} h_k & (h_k^2 - 1)/g_k \\ g_k & h_k \end{bmatrix} \begin{bmatrix} \alpha_a \\ \beta_a \end{bmatrix} = M_k \begin{bmatrix} \alpha_a \\ \beta_a \end{bmatrix}$$

$$\textit{where } h_k = y_1(\pi),\ g_k = dy_1/dt(\pi) \quad \textit{for kth cycle}$$

$$\textit{and where } y_k(t) = \alpha_k y_{1k}(t) + \beta_k y_{2k}(t)$$

The dynamics is now reduced to (infinite) matrix products:

$$M^{(N)} = \prod_{k=1}^{N} M_k(q_k, \lambda_k)$$

Highly unstable regime

$$M_k = \begin{bmatrix} h_k & (h_k^2 - 1)/g_k \\ g_k & h_k \end{bmatrix}$$

$$= h_k \begin{bmatrix} 1 & (h_k - 1/h_k)/g_k \\ g_k/h_k & 1 \end{bmatrix}$$

$$= h_k \begin{bmatrix} 1 & x_k \\ 1/x_k & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/g_k \\ 0 & 0 \end{bmatrix}$$

Highly unstable regime

$$M_k = \begin{bmatrix} h_k & (h_k^2 - 1)/g_k \\ g_k & h_k \end{bmatrix}$$

$$= h_k \begin{bmatrix} 1 & (h_k - 1/h_k)/g_k \\ g_k/h_k & 1 \end{bmatrix}$$

$$= h_k \begin{bmatrix} 1 & x_k \\ 1/x_k & 1 \end{bmatrix} - \cancel{\begin{bmatrix} 0 & 1/g_k \\ 0 & 0 \end{bmatrix}}$$

$$\left(x_k \equiv h_k / g_k \right)$$

GROWTH RATES

The growth rates for the matrix products can be broken down into two separate components, the asymptotic growth rate and the anomalous rate:

$$\gamma_\infty = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \gamma(q_k, \lambda_k) \to \langle \gamma \rangle$$

[where individual growth rates given by Floquet's Theorem]

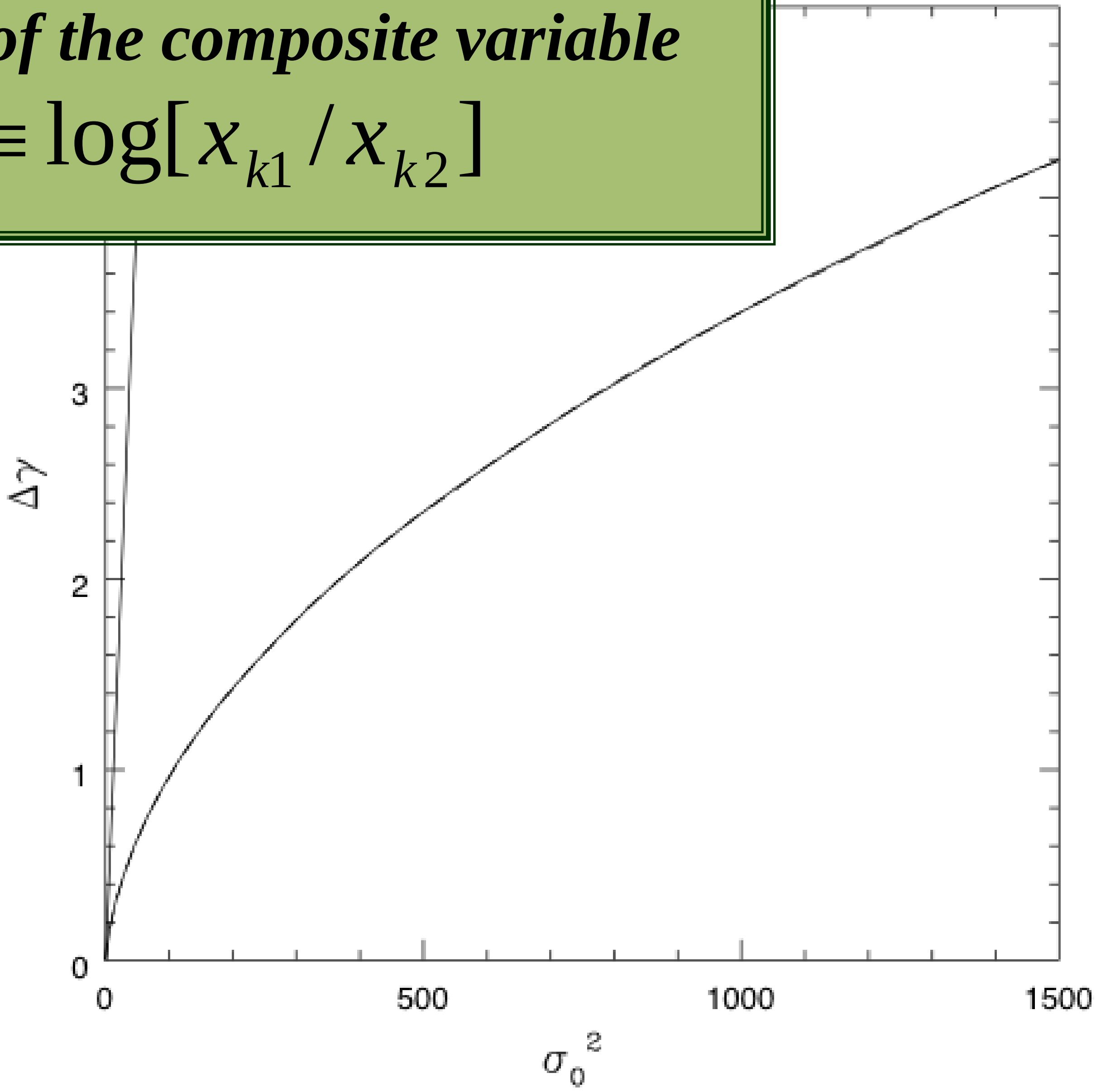
Next: take the limit of large q, i.e., unstable limit: $h >> 1$

$$\Delta\gamma = \lim_{N\to\infty} \frac{1}{\pi N} \sum_{k=1}^{N} \ln(1 + x_{k1}/x_{k2}) - \frac{\ln 2}{\pi}$$

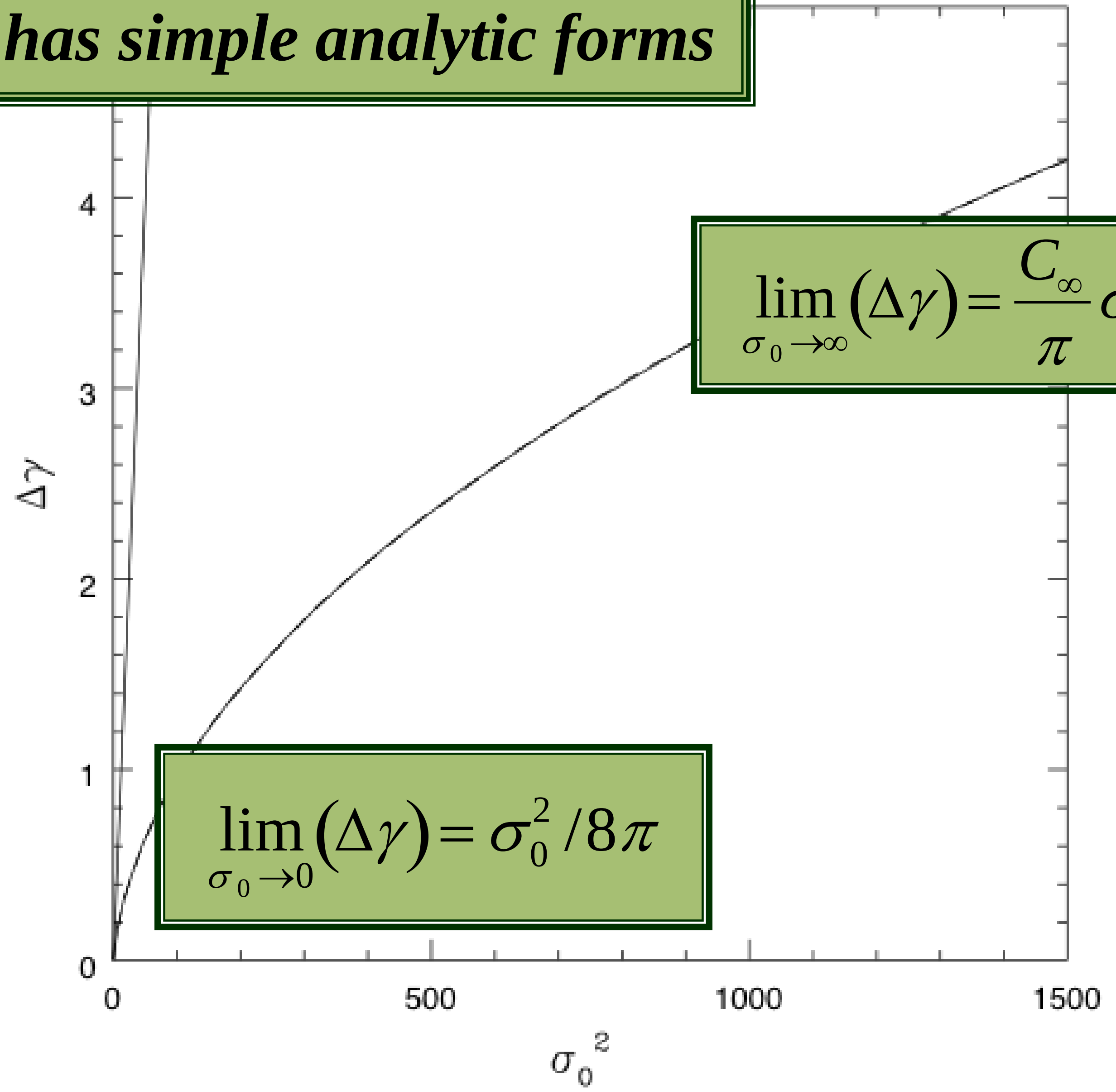
$$\textit{where}\ \ x_k \equiv h_k / g_k$$

Anomalous Growth Rate as function of the variance of the composite variable

$$\xi \equiv \log[x_{k1} / x_{k2}]$$



For asymptotic limits, the Anomalous Growth Rate has simple analytic forms



$$\lim_{\sigma_0\to\infty}(\Delta\gamma)=\frac{C_\infty}{\pi}\sigma_0$$

$$\lim_{\sigma_0\to 0}(\Delta\gamma)=\sigma_0^2/8\pi$$

Basic Theorems

- **Theorem 1: Generalized Hill's equation that is non-periodic can be transformed to the periodic case with rescaling of the parameters:** $t \to \mu_k t,\ \lambda_k \to \lambda_k/\mu_k^2,\ q_k \to q_k/\mu_k^2$

- **Theorem 2: Gives anomalous growth rate for unstable limit:**

$$\Delta\gamma = \lim_{N\to\infty}(1/\pi N)\sum_{j=1}^{N}\ln\left[1+x_{j1}/x_{j2}\right]-\ln 2/\pi$$

- **Theorem 3: Anomalous growth rate bounded by:** $\Delta\gamma \le \dfrac{\sigma_0^2}{4\pi}$

- **Theorem 4: Gives anomalous growth rate for unstable limit for forcing function having both positive and negative signs:**

$$\Delta\gamma+\frac{\ln 2}{\pi} = \lim_{N\to\infty}\frac{1}{\pi N}\left\{f_+\sum_{j=1}^{N}\ln(1+|x_{j1}/x_{j2}|)+f_-\sum_{j=1}^{N}\ln\left|1-|x_{j1}/x_{j2}|\right|\right\}$$

(Adams & Bloch 2008, 2009, 2010, 2013)

Back to Delta Function Limit

$$\frac{d^2y}{dt^2} + \left[\lambda_k + q_k\delta[t-\pi/2]\right]y = 0$$

Two limiting cases to consider:

$$(A)\ q_k >> 1$$

$$(B)\ q_k << 1$$

Matrix Elements

$$h_k = \cos\phi_k - \frac{q_k}{2\sqrt{\lambda_k}}\sin\phi_k$$

$$g_k = -\sqrt{\lambda_k}\sin\phi_k - q_k\cos^2(\phi_k/2)$$

$$\textit{where} \quad \phi_k \equiv \sqrt{\lambda_k}\,\pi$$

$$\frac{d^2y}{dt^2} + \left[\lambda_k + q_k\delta[t - \pi/2]\right]y = 0$$

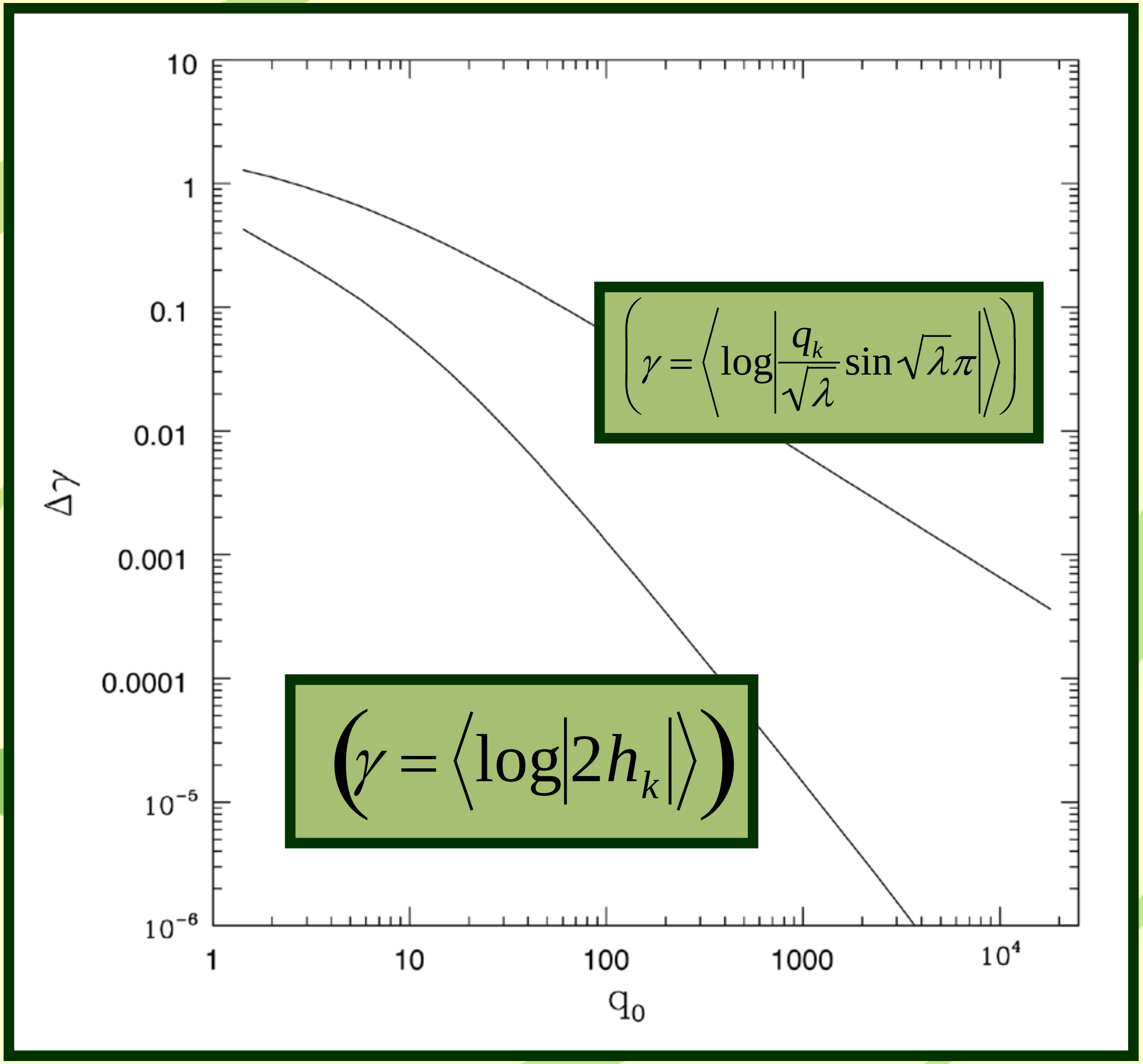
Growth rates for limiting regimes

$$q_k >> 1 \qquad \gamma = \left\langle \log\left|2h_k\right| \right\rangle + O\left(1/q_k\right)$$

$$q_k << 1 \qquad \textit{fixed} \quad \lambda$$

$$\gamma = \log\left[1 + \left\langle q_k^2 \right\rangle / 8\lambda\right] + O\left(q_k^4\right)$$

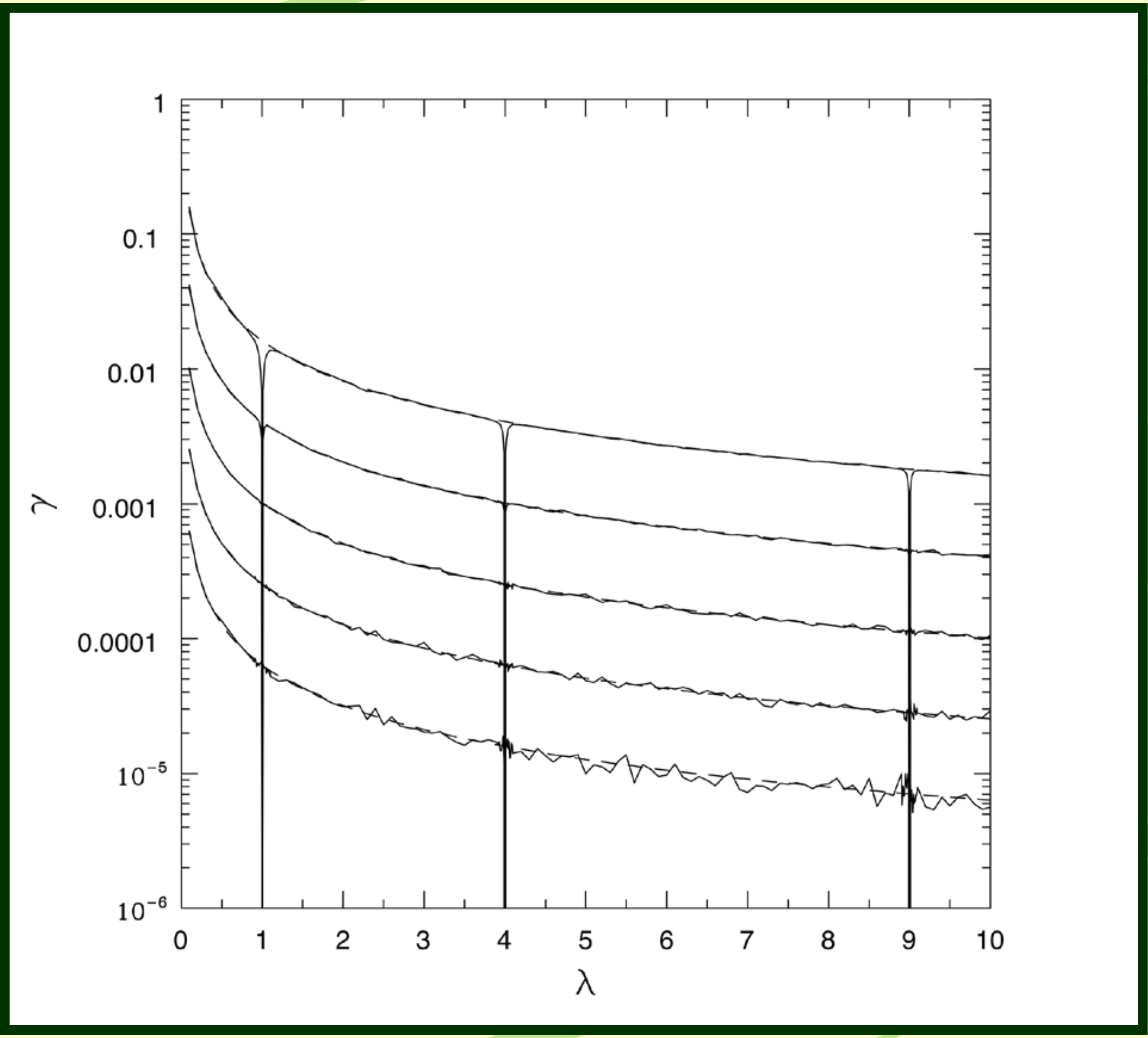
Error as function of q>>1



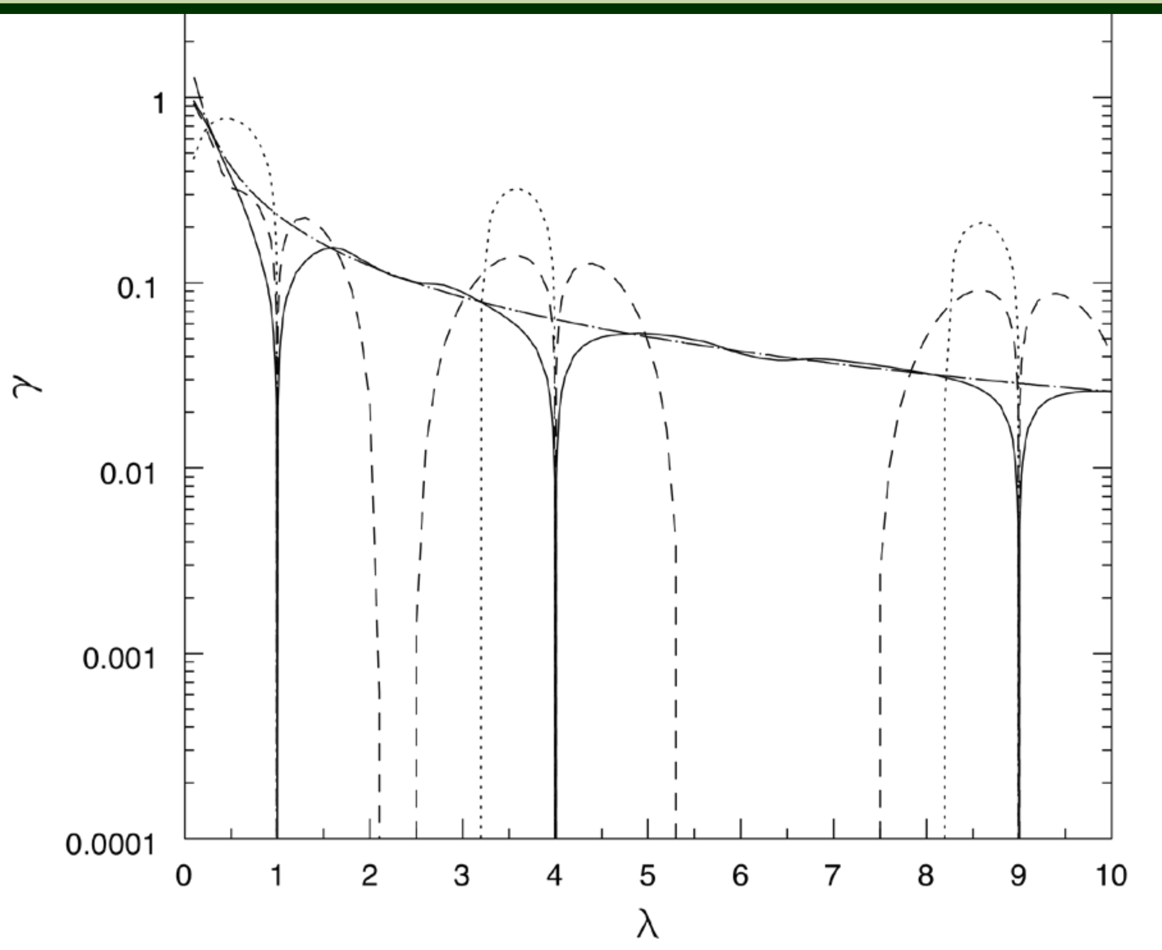
$$\left(\gamma = \left\langle \log\left|\frac{q_k}{\sqrt{\lambda}}\sin\sqrt{\lambda}\pi\right|\right\rangle\right)$$

$$\left(\gamma = \left\langle \log\left|2h_k\right|\right\rangle\right)$$

Grow Rates in q<<1 Regime

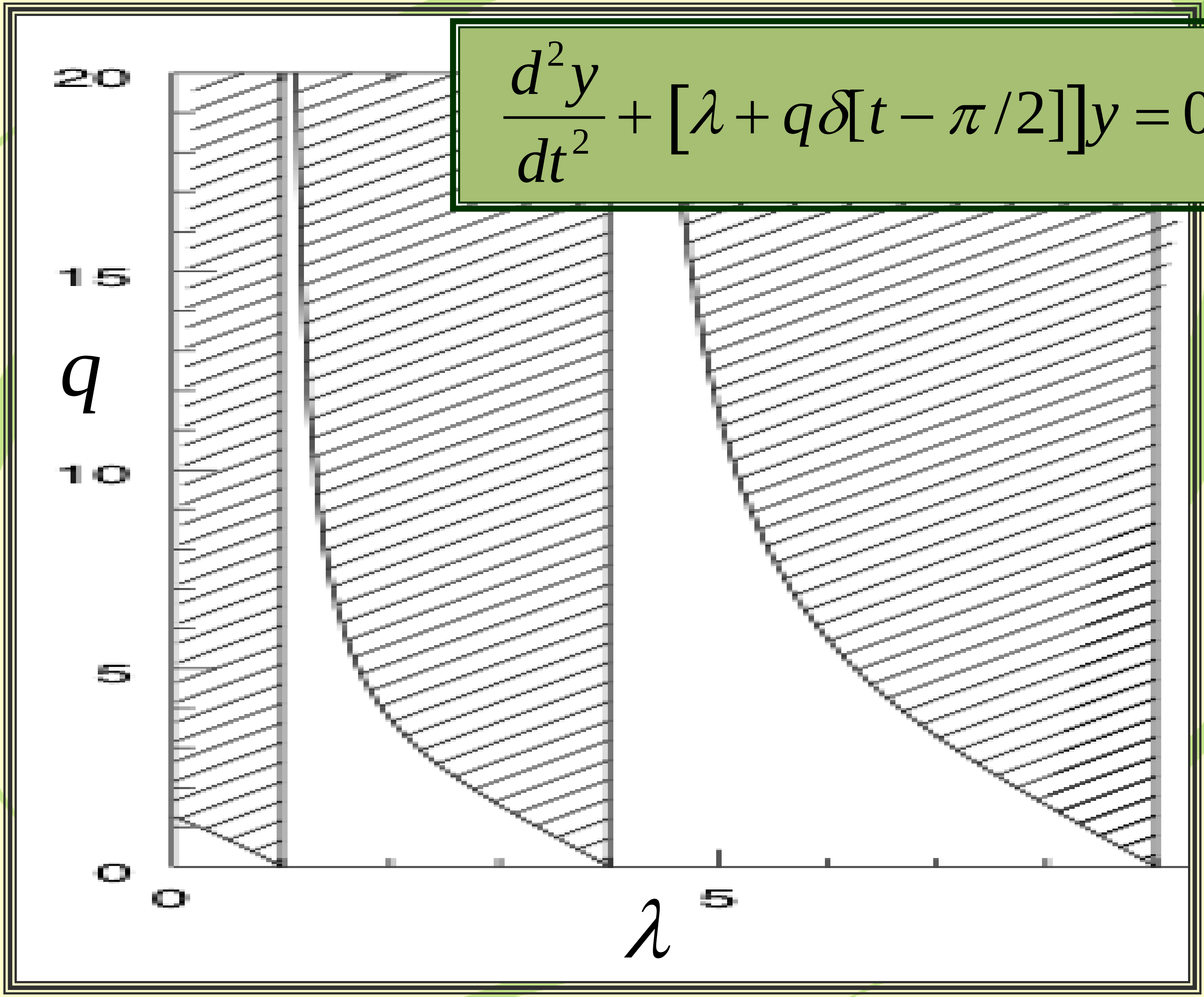


Solid: Numerical matrix multiplication
Dot-dash: Approximation for full problem
Dashed: Asymptotic growth rate
Dotted: Growth rate for constant q (non-stochastic)



Instability Strips in Delta Function Limit

$$\frac{d^2y}{dt^2}+[\lambda+q\delta[t-\pi/2]]y=0$$



Width of the Singular Zones

The approximations discussed thus far FAIL when the values of λ are near square integer values. The widths of these zones are given approximately by

$$\boxed{\delta\lambda \approx {}^{2q_k}\!/_{\pi} \quad where \quad q_k << 1}$$

Result:

$$let \quad M_k = \begin{bmatrix} 1 & x_k \\ 1/x_k & 1 \end{bmatrix} \rightarrow M_p = \begin{bmatrix} 1 & x_p \\ x_p & 1 \end{bmatrix}$$

$$and \quad let \quad x_p = p^{-s/2} \quad (where \quad p = prime)$$

$$define: \quad M_{(s)}^{(\infty)} = \prod_{p}^{\infty} M_p \quad \Rightarrow \quad M_{(4)}^{(\infty)} = \frac{3}{2\pi^2} \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

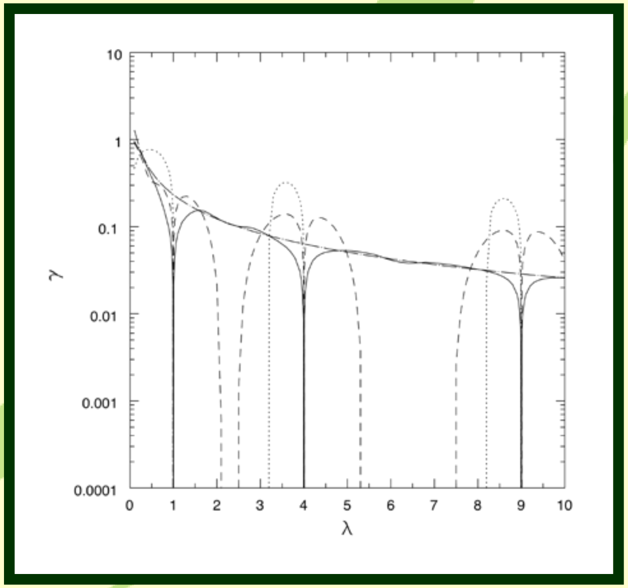
$$then: \quad eigenvalue \quad \lambda_s^{\infty} = \zeta(s/2)/\zeta(s)$$

Applied Math Results

- ***We can solve the stochastic version of Hill’s Equation (with cycle to cycle variations)***

$$\frac{d^2y}{dt^2} + \left[\lambda_k + q_k Q(\mu_k t)\right]y = 0$$

- ***The characteristic bands of Stability almost disappear for the stochastic case***



Astrophysical Applications

- ***Dark Matter Halos***: Radial orbits are unstable to perpendicular perturbations and will develop more isotropic velocity distributions (Adams et al. 2007).
- ***Reheating after Inflation in the Early Universe:*** (Kofman, Brandenberger, Linde, and many others…)
- ***Tidal Streams***: Instability will act to disperse streams; alternately, long-lived tidal streams place limits on the triaxiality of the galactic mass distribution.
- ***Galactic Structure***: Instability will affect orbits in the galactic plane – excite perpendicular motion (Binney 81)
- ***Young Stellar Clusters***: Systems are born irregular and become rounder: Instability dominates over stellar scattering. Triaxial signature has now been observed (Proszkow et al. 2009).

Mathematical Issues

- ***Transformation between distribution of the parameters in Hill's equation and distribution of the matrix elements***
- ***Relationship to Fokker-Planck treatments***
- ***Grow rates for the generalized problem***
- ***Generalization to other periodic (and stochastic) differential equations (e.g., stochastic pendulum)***
- ***Relationship to random matrix theory***

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- ***Orbits in Extended Mass Distributions, Adams & Bloch, 2005, ApJ, 629, 204***
- ***Ultimate Halo Mass in LCDM Cosmology, Busha, Evrard, Adams, 2005, MNRAS***
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