The matrix methods in the stability theory of stellar systems and the problem of bar formation

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Contents

"Incidentally, if you are looking for a good problem,
the exact details of how the arms are formed
and what determines the shapes of these galaxies
has not been worked out."

R.P.Feynman, 1963

formulated about 50 years ago, the problem remains unsolved.

Recognized mechanisms of the spiral structure formation:

- Tidal effects
- Minor merging & gas infall
- Secular evolution – instability

Study of stability of stellar systems has 2 major goals:

- Explanation of the observable spiral patterns of disk galaxies
- Search for appropriate stationary models
  (in equilibrium and stable)

In numerical experiments much more sophisticated systems, with complex geometry, gas and stellar content, etc. can be studied. However, there are some drawbacks:

- Uncertainty of the results
- Difficult interpretation of instability
- Numerical effects

Thus, various approaches are important.

There are a few types of systems that allow analytical approach to study of stability:

- Homogeneous systems (stellar and fluid) — *Jeans*
- Short-wave WKB theory — *Lin, Shu 1964, 1966; Kalnajs 1965*
- Some special models:
  - isotropic spheres — *Antonov 1960, 1962*
  - flat homogeneous layer — *Antonov 1971, Kalnajs 1973*
  - solid body rotation — *1970+*
    - stellar disk — *V.Polyachenko, Shukhman; Kalnajs*
    - fluid disk — *Hunter*
    - sphere with circular orbits — *Mikhailovsky, Fridman, Eppelbaum*
    - sphere with elliptical orbits — *V.Polyachenko, Shukhman*
    - Ellipsoid (oblate and triaxial) — *Morozov, V.Polyachenko, Shukhman*

Matrix methods search for exponentially growing perturbations of density and potential in stellar disks leading to formation of a spiral pattern, but does not explain underlying physics

The biorthonormal potential-density basis *Kalnajs 1977*

$$\nabla^2\Phi_\alpha = 4\pi G\rho_\alpha$$

$$-\int \mathrm{d}^3\mathbf{x}\Phi^*_\alpha(\mathbf{x})\rho_\beta(\mathbf{x}) = \delta_{\alpha\beta}$$

The matrix equation ($\omega = m\Omega_\mathrm{p} + i\gamma$, m is the azimuthal number)

$$\mathrm{Det}\left\|\delta^{\alpha\beta} - M^{\alpha\beta}(\omega)\right\| = 0$$

$$M^{\alpha\beta}(\omega) = (2\pi)^3\sum_{\mathbf{m}}\int d^3\mathbf{J} f_0(\mathbf{J})\mathbf{m}\cdot\frac{\partial}{\partial\mathbf{J}}\left[\frac{\Phi^*_{\alpha,\mathbf{m}}(\mathbf{J})\Phi_{\beta,\mathbf{m}}(\mathbf{J})}{\mathbf{m}\cdot\Omega(\mathbf{J}) - \omega}\right]$$

$$\Phi_\alpha[\mathbf{x}(\boldsymbol{\theta},\mathbf{J})] = \sum_{\mathbf{m}}\Phi_{\alpha,\mathbf{m}}(\mathbf{J})e^{i\mathbf{m}\cdot\boldsymbol{\theta}}$$

Zang (1976), Kalnajs (1978), Sawamura (1988), Vauterin & Dejonghe (1996), Pichon & Cannon (1997), Evans & Read (1998)

- Elongation and capturing orbits in systems with slowly rising rotation curves — *Lynden-Bell 1979*
- Swing amplification — *A.Toomre 1981*
- WASER II mechanism — *G.Bertin 1983*

Toomre 1981

WHAT AMPLIFIES THE SPIRALS?

Alar Toomre

Massachusetts Institute of Technology

It seems clear now that the spiral structure of galaxies is a complex riddle without any unique and tidy answer. At most, one might perhaps still hope to blame it all on differential rotation, one way or another. Beyond this quasi-platitude, however, we know that the logical paths diverge soon and deservedly toward such separate themes as global instabilities, stochastic spirals, and also the spectacular shock patterns that can arise in shearing gas disks when forced by bars, to cite just the three areas which have progressed most vigorously as of late.

Any reviewer of this knotty subject faces a breadth-vs-depth dilemma: Ought he really to try and grapple with several of these diverse topics? Or should one instead focus on just one or two? Not too long ago I took the first tack in an article (Toomre 1977) that ran to 40+ pages. Here I opt for the opposite course, and shall dwell mainly on a single topic from the wave mechanics of a disk galaxy. I wish to reacquaint you with a neat old phenomenon that I have lately taken to calling swing amplification.

At first, this choice of emphasis may strike you as unusually biased and narrow. I rather hope it does — for then I can teach you something — but I assure you it is neither. On the contrary, here seems to be a rare opportunity for a reviewer to point to an almost forgotten process and to urge "Wait, try this one again!". But why? For one thing, I believe this amplification will help us all, at long last, to understand (and to repair) many of the global instabilities that have plagued our subject for the past decade. And as you will see, this same powerful phenomenon also makes it astonishingly plausible that some of the finest "normal" spirals, like M51 and M81, may in fact be just beautiful tidal transients.

One extra reason makes this topic singularly fitting. Though not yet by that name, swing amplification in a fairly small-scale or local context was first noticed right here in old Cambridge, in late 1963, by Goldreich & Lynden-Bell (1965, hereinafter GLB). Now let me illustrate at once what it can do in the large, and why it deserves a lot more attention than it has received.

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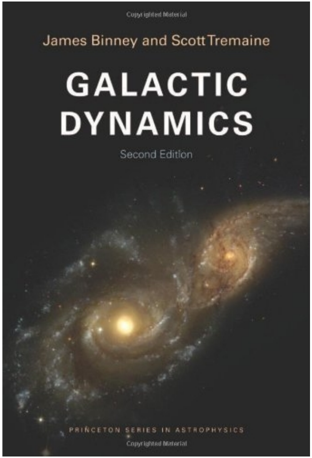
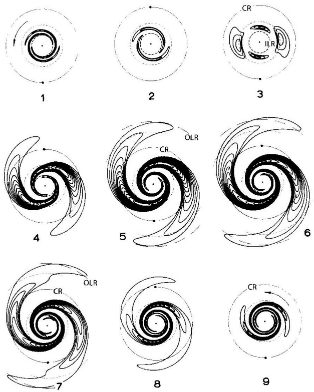
Binney&Tremaine 2008



Swing amplification



- CR region
- Feedback through the center

Spiral & bar formation

514 *Chapter 6: Disk Dynamics and Spiral Structure*

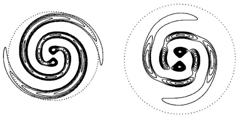


Figure 6.23 Shapes of unstable modes in a Gaussian disk (Toomre 1981). Left: no rigid halo. Right: one-third of the central force arises from a rigid halo. The growth rate is 17% of the pattern speed in the first case but only 3% of the pattern speed in the second. This calculation used a cold disk and softened gravity to model a disk with non-zero Q (see Problem 6.3). Dots mark the corotation circle. From Toomre (1981), © Cambridge University Press 1981. Reprinted by permission of Cambridge University Press.

- spirals – high amplification
- bars – low amplification
- bars – lumpy structure (superposition of tightly wound leading and trailing spirals)

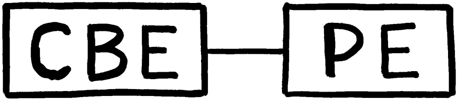
Origin of bars

Dynamics of collisionless stellar systems is governed by collissionless Boltzmann equation (CBE)

$$\frac{\partial f}{\partial t} + [f, H] = 0 \tag{1}$$

and Poisson equation

$$\Phi(\mathbf{x}, t) = -G \int \mathrm{d}^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \tag{2}$$

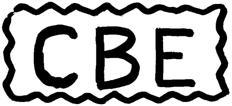


In systems with periodic and quasiperiodic motions it is favorable to use action-angle variables $(\mathbf{J},\boldsymbol{\theta})$, so CBE takes the form

$$\frac{\partial f}{\partial t}+\sum_k\left(\frac{\partial H}{\partial J_k}\frac{\partial f}{\partial \theta_k}-\frac{\partial H}{\partial \theta_k}\frac{\partial f}{\partial J_k}\right)=0 \tag{3}$$

Linearization $f(\mathbf{J},\boldsymbol{\theta},t)=f_0(\mathbf{J})+\sum_{\mathbf{m}} f_{\mathbf{m}}^{(1)}(\mathbf{J})e^{i\mathbf{m}\cdot\boldsymbol{\theta}-i\omega t}$, $H=H_0(\mathbf{J})+\Phi_1(\mathbf{J},\boldsymbol{\theta},t)$ gives

$$\sum_{\mathbf{m}}(\omega-\mathbf{m}\cdot\boldsymbol{\Omega})f_{\mathbf{m}}^{(1)}(\mathbf{J})e^{i\mathbf{m}\cdot\boldsymbol{\theta}-i\omega t}=i\sum_k\frac{\partial \Phi_1}{\partial \theta_k}\frac{\partial f_0}{\partial J_k} \tag{4}$$



The Poisson equation is linear, so

$$\Phi_1(\mathbf{x}[\mathbf{J},\boldsymbol{\theta}],t) = -G\int \mathrm{d}\mathbf{J}'\mathrm{d}\boldsymbol{\theta}'\,\frac{f^{(1)}_{\mathbf{m}'}(\mathbf{J}')e^{i\mathbf{m}'\cdot\boldsymbol{\theta}'-i\omega t}}{|\mathbf{x}[\mathbf{J},\boldsymbol{\theta}]-\mathbf{x}[\mathbf{J}',\boldsymbol{\theta}']|} \tag{5}$$

Formally, $\Phi_1(\mathbf{J},\boldsymbol{\theta},t) = \sum_{\mathbf{m}} \tilde{\Phi}^{(1)}_{\mathbf{m}}(\mathbf{J})e^{i\mathbf{m}\cdot\boldsymbol{\theta}-i\omega t}$, thus

$$\tilde{\Phi}^{(1)}_{\mathbf{m}}(\mathbf{J}) = -G\int \mathrm{d}\mathbf{J}'\sum_{\mathbf{m}'}\mathcal{K}_{\mathbf{m},\mathbf{m}'}(\mathbf{J},\mathbf{J}')f^{(1)}_{\mathbf{m}'}(\mathbf{J}') \tag{6}$$

where

$$\mathcal{K}_{\mathbf{m},\mathbf{m}'}(\mathbf{J},\mathbf{J}') = -\frac{1}{(2\pi)^N}\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}'\,\frac{e^{i\mathbf{m}'\cdot\boldsymbol{\theta}'-i\mathbf{m}\cdot\boldsymbol{\theta}}}{|\mathbf{x}[\mathbf{J},\boldsymbol{\theta}]-\mathbf{x}[\mathbf{J}',\boldsymbol{\theta}']|} \tag{7}$$

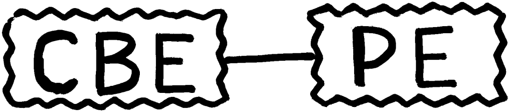
Collecting CBE and Poisson equations, we obtain

$$(\omega - \mathbf{m}\cdot\mathbf{\Omega}) f_{\mathbf{m}}^{(1)}(\mathbf{J}) = -\sum_k m_k \tilde{\Phi}_{\mathbf{m}}^{(1)}(\mathbf{J}) \frac{\partial f_0}{\partial J_k} \qquad (4)$$

and

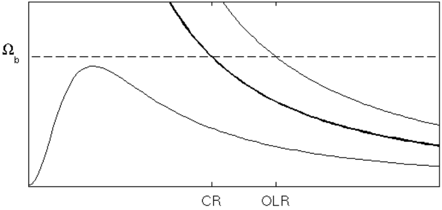
$$\tilde{\Phi}_{\mathbf{m}}^{(1)}(\mathbf{J}) = -G \int \mathrm{d}\mathbf{J}' \mathcal{K}_{\mathbf{m},\mathbf{m}'}(\mathbf{J},\mathbf{J}') f_{\mathbf{m}'}^{(1)}(\mathbf{J}') \qquad (6)$$

Substituting (6) into (4) gives a linear eigenvalue problem for ω for any integrable system $H_0(\mathbf{J})$.



- Angular velocity $\Omega(R) = V(R)/R$
- Curves $\Omega(R) \pm \kappa(R)/m$ for Lindblad resonances with radial oscillations ($\kappa(R)$ is epicyclic frequency, $m = 2$)
- Bar pattern speeds are slightly above the maximum of the precession rate $\Omega_{\rm pr}(R) \equiv \Omega(R) - \kappa(R)/2$

... e.g., Combes & Elmegreen 1993, Lynden-Bell 1996



In the center:

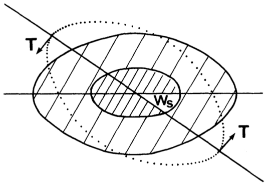
$$|\Omega_{\rm pr}(R) - \Omega_{\rm b}| \ll |\Omega(R) - \Omega_{\rm b}| \quad (8)$$

"On the mechanism that structures galaxies" *Lynden-Bell 1979*

$$|\Omega_{\rm pr}(R) - \Omega_{\rm b}| \ll |\Omega(R) - \Omega_{\rm b}| \qquad (8)$$

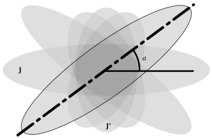
Average out the radial motion of stars along the orbits and consider only precessional motion of the orbits (wires).

An oval orbit (dots) is attracted by the emerging bar (shaded) and can be captured. The bar pattern speed is then a compromize between precession rates of captured orbits.



Observations: bars do consist of "two-lobe one turn" orbits, but bar pattern speeds slightly exceed maximum precession rate.

Lynden-Bell 1996



A DF for wires

$$f(\mathbf{J}, \alpha, t)$$

Actions $(\mathbf{J})$ fix the orbit's shape, α is the azimuth of the line of apsides.

The linear eigenvalue problem for ω:

$$(\omega - m\Omega_{\rm pr}) f^{(1)}(\mathbf{J}) = -m\tilde{\Phi}^{(1)}(\mathbf{J}) \frac{\partial f_0}{\partial L}$$

$$\tilde{\Phi}^{(1)}(\mathbf{J}) = -G \int \mathrm{d}\mathbf{J}' \mathcal{K}(\mathbf{J}, \mathbf{J}') f^{(1)}(\mathbf{J}')$$

This is a "core" of the exact equation for ω, that includes all resonances. It can be obtained rigorously from the exact equation by averaging over the orbital motion.

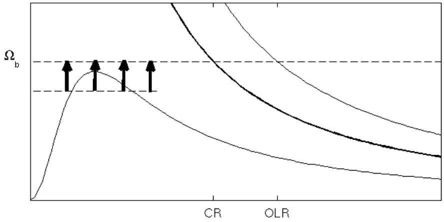
Polyachenko: MNRAS 2004, 2005

$$(\omega - m\Omega_{\rm pr}(\mathbf{J}))f^{(1)}(\mathbf{J}) = Gm\frac{\partial f_0(\mathbf{J})}{\partial L}\int \mathrm{d}\mathbf{J}'\mathcal{K}(\mathbf{J},\mathbf{J}')f^{(1)}(\mathbf{J}') \qquad (9)$$

Pattern speed $\Omega_{\rm p} = {\rm Re}\,\omega/m$ is a compromise between precession rates biased by selfgravity.

$\mathcal{K}(\mathbf{J},\mathbf{J}')$ is proportional to a torque between orbits of shapes $\mathbf{J}$ and $\mathbf{J}'$.

Normally $\partial f_0/\partial L > 0$, thus the bias is positive.



Directly from (9), $\partial f_0/\partial L > 0$

$$(\mathrm{Im}\,\omega)L_m = 0 \quad , \quad L_m \equiv -\int \mathrm{d}\mathbf{J}\,\frac{|f^{(1)}(\mathbf{J})|^2}{\partial f_0/\partial L} < 0 \qquad (10)$$

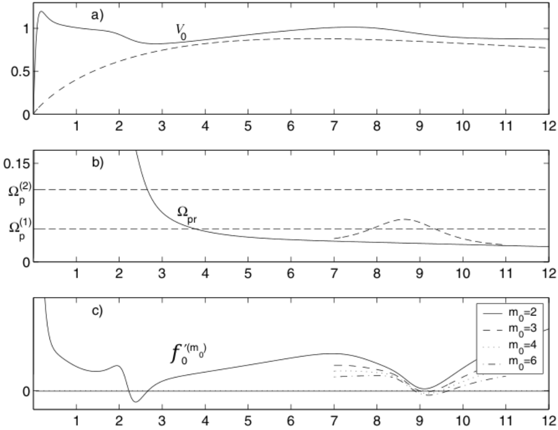
L_m is the angular momentum of the mode, the growth rate $\gamma \equiv \mathrm{Im}\,\omega = 0$.

- The "core" equation estimates the pattern speed only
- Self-gravity generates a pattern that rotates rigidly even faster than the fastest precession rate in the disc
- Growth of the pattern amplitude is due to interaction with stars on resonances (CR, OLR, etc.) and can be calculated if $\gamma \ll \Omega_{\mathrm{p}}$

Lynden-Bell, Kalnajs 1972

$\partial f_0 / \partial L < 0$ is possible: additional unstable modes may occur

- dynamically "hot" center
- pecularities of velocity curves



Example: comparison of theoretical modes ("core" equation) and N-body simulations

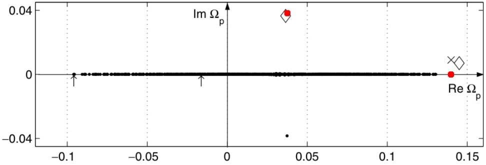


Figure 5. The spectrum of complex pattern speeds (filled circles) computed as the eigenvalues of problem (16), for the model $m=6, \beta=3, q=1, J_c=0.6$ of Athanassoula & Sellwood (1986). The cross is the complex pattern speed,

Polyachenko, 2004

The exact eigenvalue equation looks similar, but involves different resonances:

$$[\omega - n\Omega_1(\mathbf{J}) - m\Omega_2(\mathbf{J})] f_n^{(1)}(\mathbf{J}) = G f_0^{\prime (n)} \int \mathrm{d}\mathbf{J}' \sum_{n'} \mathcal{K}_{n,n'}(\mathbf{J}, \mathbf{J}') f_n^{(1)}(\mathbf{J}') \qquad (11)$$

where

$$m \frac{\partial f_0(\mathbf{J})}{\partial L} \quad \rightarrow \quad f_0^{\prime (n)} \equiv n \frac{\partial f_0(\mathbf{J})}{\partial J_R} + m \frac{\partial f_0(\mathbf{J})}{\partial L}$$

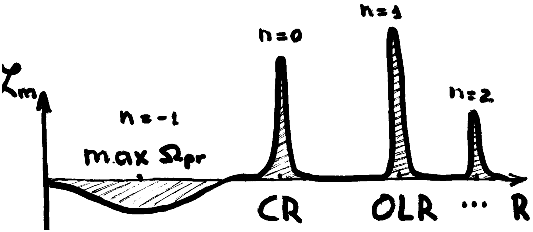
From equation (11)

$$(\mathrm{Im}\,\omega) L_m = 0 \quad , \quad L_m \equiv -\sum_n \int \mathrm{d}\mathbf{J} \frac{|f_n^{(1)}(\mathbf{J})|^2}{f_0^{(n)}(\mathbf{J})} \qquad (12)$$

For unstable modes, $\mathrm{Im}\,\omega > 0$, thus angular momentum of the wave $L_m = 0$ (angular momentum conservation).

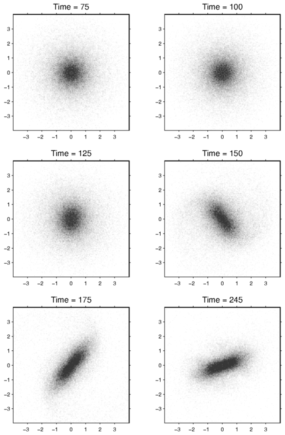
$$L_m \equiv -\sum_n \int \mathrm{d}\mathbf{J} \frac{|f_n^{(1)}(\mathbf{J})|^2}{f_0^{(n)}(\mathbf{J})} = 0$$

Schematically, angular momentum density distribution of bar modes over radius looks as follows:



Angular momentum is redistributed from the central part (region of precession rate hump) to resonances (corotation, outer Lindblad, etc.). The same with energy.

Kuzmin-Toomre SG disk with retrograde orbits $Q \approx 1.5$



- Usual PM planar scheme
- Gravity softening $r^{-1} \to (r^2 + \varepsilon^2)^{-1/2}$
- Quiet start *Athanassoula, Sellwood 1986*

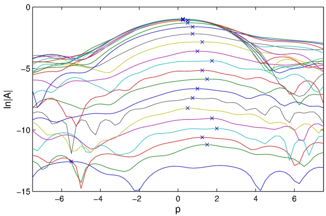
Visually, one can distinguish three stages

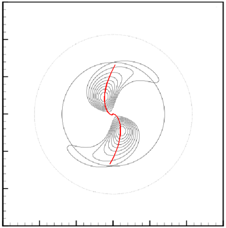
- Axisymmetric, $t < 120$
- Bar formation, $120 < t < 150$
- Non-linear bar, $t > 150$

Fourier analysis and expansion in logarithmic spirals

$$A_m(p,t) = \int_0^{R_{max}} \int_0^{2\pi} \Sigma(r,\varphi,t) \exp(-i[m\varphi + p \ln r]) r \mathrm{d}r \mathrm{d}\varphi \qquad (13)$$

Pitch angle $\alpha = \mathrm{arctg}(m/p)$, p is spirality (trailing $p > 0$, bars $p = 0$).
Bisymmetric perturbations: $m = 2$.



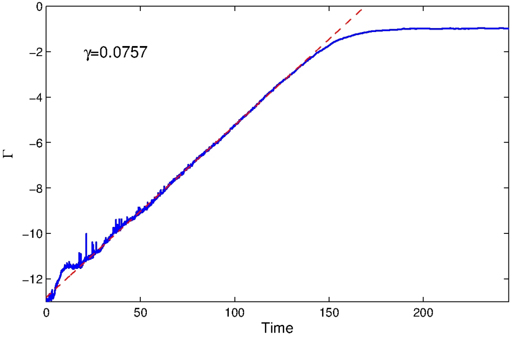


Jalali, Hunter, 2005 (JH05)

No sign of tightly wound leading and trailing spirals

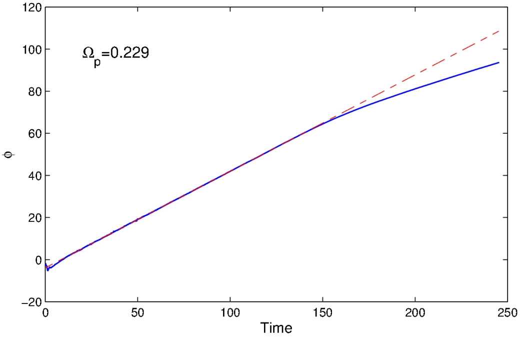
Exponential growth of the amplitude in the linear regime

$$\Gamma(t) = \max_p \ln |A_2(p, t)|$$

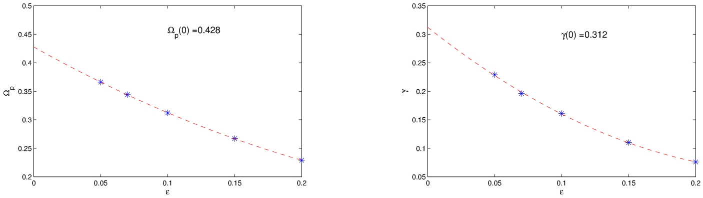


Constant pattern speed in the linear regime

$$\Omega_p^{\varepsilon} = \frac{1}{m}\frac{d\phi}{dt}$$

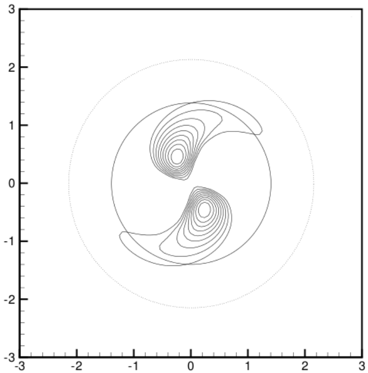


Extrapolation to $\varepsilon \to 0$ and comparison with JH05



ε	0.20	0.15	0.10	0.07	0.05	0	JH1	JH2
Ω_p	0.229	0.267	0.314	0.344	0.366	0.428	0.445	0.294
γ	0.076	0.110	0.161	0.196	0.229	0.312	0.308	0.109

Thus, the detected mode is the most unstable mode found by JH05, i.e. the mode with the largest growth rate



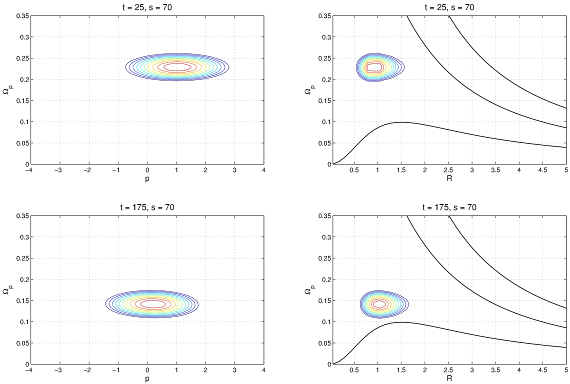
Power spectrum “spirality–pattern speed”

$$\mathcal{P}_t^s(p,\Omega_p) = \frac{1}{2\pi}\left| \int_t^{t+s} \frac{A_2(p,t')}{\exp\Gamma(t')} e^{im\Omega_p t'} H_t^s(t') dt' \right|^2$$

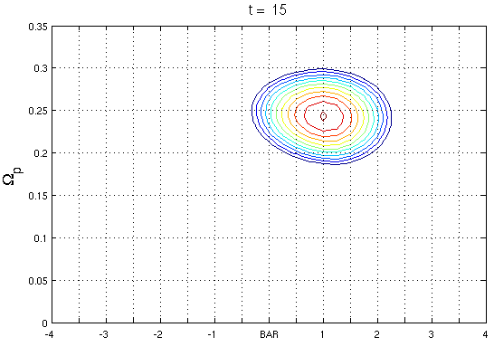
Power spectrum “radius–pattern speed”

$$\mathcal{R}_t^s(R,\Omega_p) = \frac{1}{2\pi}\left| \int_t^{t+s} \frac{\tilde{A}_2(R,t')}{\exp\Gamma(t')} e^{im\Omega_p t'} H_t^s(t') dt' \right|^2$$

Initial and final power spectra



Time evolution of “spirality–pattern speed” spectrum



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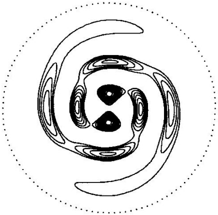
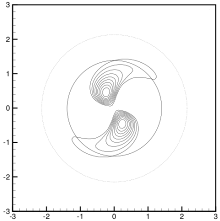
Swing amplification



- "lumpy" structure
- low growth rate

Normal bar mode



- one or few antinodes
- highest growth rate
- single growing mode

Several mechanisms of bar formation

The qualitative (kinetic) description, $\gamma \ll \Omega_{\rm p}$:

- A "body" of the bar is formed by orbits in the central part of the disk. It carries negative angular momentum and energy.
- The bar drags stars on resonances (CR, OLR, etc.) and gives them angular momentum. The mechanism is similar to inverse Landau damping. So, not only the region of corotation responsible for bar formation.
- Stabilizing effects of heavy halo and centrally concentrated mass is naturally explained by insufficient role of selfgravity to establish a mode with $\Omega_{\rm p} > \max(\Omega_{\rm pr})$

In general $\gamma \lesssim \Omega_{\rm p}$, bar-forming instability is no longer strictly kinetic.

Summary on the matrix method:

- can be applied to any integrable system $H_0(\mathbf{J})$.
- linear with respect to the unknown frequency ω. This greately simplifies the search for solutions, no risk of losing the roots.
- Exact equations for spheres and disks are available
- the "core" equation can be figured out
- the components of the equation can be interpreted in simple physical terms

Works that involve the matrix method:

- Thin disks: P. 2004, 2005, P.,P.,Sh. 2007-2010
  - Qualitative description of bar using slow mode approach ("core" equation)
  - Features of the spiral structure in Milky way type galaxies
  - Toomre-Zang model of a stellar disk
  - Gravitation loss-cone instability
  - Radial orbit instability
- Spheres: P., V.Polyachenko, Shukhman 2007-2012
  - Gravitation loss-cone instability
  - Radial orbit instability
  - Generalized polytropes

Problems and further studies

- Further comparison with SA & WASER II
- Accuracy of the method
- Role of resonances and of the feedback
- Extensive comparison with the known models and N-body
- Application to Milky Way type galaxies

THANK YOU!