Generalized Lin–Shu density-wave theory

Three-dimensional and spatially inhomogeneous galactic disk

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June 24–28, 2013
The Lin–Shu Symposium, Tsinghua Univ.

אוניברסיטת-בן גוריון בנגב

המחלקה לפיסיקה

Spiral Galaxies: Face-on View of M74

Astronomy Picture of the Day

Discover the cosmos! Each day a different image or photograph of our fascinating universe is featured, along with a brief explanation written by a professional astronomer.

2001 October 4

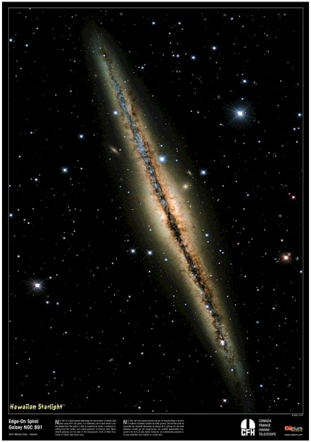
M74: The Perfect Spiral
Credit: Gemini Observatory, GMOS Team

Explanation: If not perfect, then this spiral galaxy is at least one of the most photogenic. An island universe of about 100 billion stars, 30 million light-years away toward the constellation Pisces, NGC 628 or M74 presents a gorgeous face-on view to earthbound astronomers. Classified as an Sc galaxy, the grand design of M74's graceful spiral arms traced by bright blue star clusters and dark cosmic dust lanes, is similar in many respects to our own home galaxy, the Milky Way. Recorded with a 28 million pixel detector array, this impressive image celebrates first light for the Gemini Multi-Object Spectrograph (GMOS), a state-of-the-art instrument now operational at the 8-meter Gemini North telescope. The Gemini North Observatory gazes into the skies above Mauna Kea, Hawaii, USA, while its twin observatory, Gemini South, is scheduled to begin operations later this year from Cerro Pachón in

Lin & Shu (1964 ApJ): "Grand-design" spirals

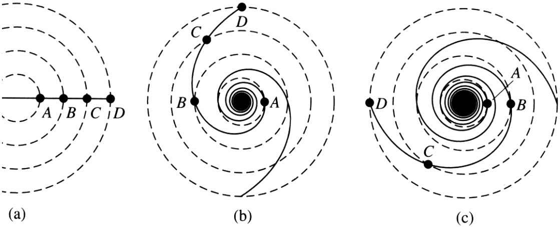
Observations

Spiral Galaxies: Edge-on View of NGC891



Observations

Spiral Structure: Winding Problem



- **Naive idea: Material arms**
- **Differential rotation of the disk!**
- **After only a few periods: arms wound too tightly to be observed!**

Observations

Original Lin–Shu theory: Grand-design Spirals

- **Differentially rotating disks of stars and gaseous clouds**
- **Number of stars:** $N \sim 10^{11}$
- **Thin systems: the width/radius ratio is** $\lesssim 0.1$
- **Gaseous subsystem: the gas/star ratio is** $\lesssim 0.1$
- **Spiral structure consists of two "grand-design" arms**
- **Lin & Shu: (i) Explain such a regular long-term** $\sim 10^9$ **yr spiral pattern**
- **Lin & Shu: (ii) Explain the concentration of centers of star formation in the spiral arms**

Spiral Galaxies: Modern Observations

- **Ring-spiral** ~ 1 **kpc structure**
- **The multi-armed** $(m > 2)$ **spiral structure**
- **The non-axially symmetric spiral pattern**
- **Actual rotating galaxies are not infinitesimally thin**
- **Actual rotating galaxies are spatially inhomogeneous**

Theory

- Lin & Shu (1964 ApJ), Lin & Shu (1966 PNAS), Lin, Yuan & Shu (1969 ApJ), Yuan (1969 ApJ), Shu (1970 ApJ): (i) quasi-stationary (ii) axisymmetric (circular) density waves propagating in a (iii) two-dimensional and (iv) homogeneous disk
- In our work: the analytical solution of the self-consistent system of the gas-dynamic equations and the Poisson equation describing the stability of a galactic disk
- Unlike Lin, Shu, Yuan and others
  we consider the gravitationally (i) unstable (amplitude-growing) (ii) both axisymmetric and non-axisymmetric density waves developing in (iii) three-dimensional and (iv) spatially inhomogeneous disk

Theory

- **The stellar systems act collisionlessly over the Hubble time**
- **Since the collisional effects are negligible, the motion equations are the Euler equations**

$$\frac{\partial v_r}{\partial t} + (\mathbf{v}\nabla)v_r - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial \Phi}{\partial r} \quad (1)$$

$$\frac{\partial v_\varphi}{\partial t} + (\mathbf{v}\nabla)v_\varphi + \frac{v_\varphi v_r}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} - \frac{1}{r}\frac{\partial \Phi}{\partial \varphi} \quad (2)$$

$$\frac{\partial v_z}{\partial t} + (\mathbf{v}\nabla)v_z = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \frac{\partial \Phi}{\partial z} \quad (3)$$

$$(\mathbf{v}\nabla) = v_r\frac{\partial}{\partial r} + \frac{v_\varphi}{r}\frac{\partial}{\partial \varphi} + v_z\frac{\partial}{\partial z} \quad (4)$$

- r, φ, z **are the galactocentric cylindrical coordinates and the axis of the disk rotation is taken oriented along the** z**-axis,** $\rho(\mathbf{r}, t)$ **is the density,** $\mathbf{v}(\mathbf{r}, t)$ **is the velocity,** $P(\mathbf{r}, t)$ **is the pressure and** $\Phi(\mathbf{r}, t)$ **is the gravitational potential**
- **The motion equations are supplemented with the continuity equation**

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\varphi)}{\partial \varphi} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad (5)$$

and the Poisson equation for the gravitational potential

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho \qquad (6)$$

- **The isothermal and isotropic equation of state** $P = c_s^2 \rho$**, where** c_s **is the sound speed**

- **Equations (1)–(6) are the system of partial differential equations with variable coefficients**

$$\rho = \rho(r, z)$$

- **Simple two-dimensional disk: Generally, the solution is**

$$X(\mathbf{r}, t) = X_0(r) + \sum_{\mathbf{k}} X_{1,\mathbf{k}}(r) \exp(\mathrm{i}m\varphi - \mathrm{i}\omega t) \qquad (7)$$

m is the azimuthal mode number, $\omega = \Re\omega + \mathrm{i}\Im\omega$ is the complex wavefrequency and $\mathrm{i} = \sqrt{-1}$

- **As a result we have**

$$\frac{\mathrm{d}^2\Phi_1}{\mathrm{d}r^2} + \aleph(r)\Phi_1(r) = 0 \;\Rightarrow \text{The solution?} \qquad (8)$$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{2m}{h^2}\left[W - V(x)\right]\Psi = 0 \;\Rightarrow \text{One} - \text{dimensional SchrEq}$$

- **To find an approximate method of solving Eqs (1)–(6)!**

Theory

- We are concerned with the growth or decay of small perturbations from an equilibrium state
- Equations (1)–(6) are the system of partial differential equations
- In the linear approximation, the time-dependent $\rho(\mathbf{r}, t)$, $\Phi(\mathbf{r}, t)$, $P(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ of a disk are represented by

$$X(\mathbf{r}, t) = X_0(r, z) + \sum_{\mathbf{k}} X_{1,\mathbf{k}}(r, \varphi, z, t)$$

- $X_0 \neq X_0(t)$ describes the basic flow and $|X_{1,\mathbf{k}}/X_0| \ll 1$ describe small perturbations
- An approximate method of solving Eqs (1)–(6): the WKB (Wentzel-Kramers-Brillouin) method in quantum mechanics

Theory

- **The WKB method can be applied when**

$$\lambda/L \ll 1$$

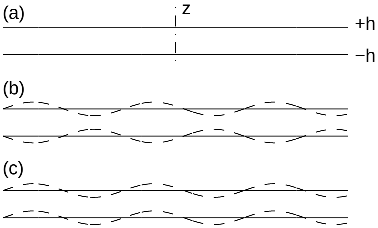
  λ **is the radial wavelength and** L **is the scale of radial inhomogeneity**

- **Using horizontally local WKB method, one may determine solutions in the form of normal modes**

$$X_1(\mathbf{r}, t) = \Re\left[\widetilde{X}(z)\exp\left(\mathrm{i}k_r r + \mathrm{i}m\varphi - \mathrm{i}\omega t\right)\right] \tag{9}$$

- $\tilde{X}(z)$ **is the real amplitude,** $k_r r$ **is a large quantity (**$k_r r \gg 1$**) and** $k_r = \mathrm{const}$ **is the real radial wavenumber (**$k_r = 2\pi/\lambda$**)**
- **If** $\Im\omega > 0$**: instability,** $X_1 \propto e^{\Im\omega \cdot t}$

Theory



Sketch of perturbations of a three-dimensional galactic disk

- **In (a) a section of the disk is shown edge-on**
- **In (b) an even compression Lin–Shu-type perturbation (the dashed line). For such longitudinal perturbations, the equatorial $z = 0$ plane remains unchanged**
- **In (c) an odd bending-type perturbation. These transverse perturbations do distort the $z = 0$ plane**

Equilibrium

- **The unperturbed disk has velocity $\mathbf{v}_0 = (0, r\Omega, 0)$, where $\Omega(r)$ is the angular rotational velocity**
- **Planar equilibrium**

$$r\Omega^2 = \frac{\partial \Phi_0}{\partial r} + \frac{c_s^2}{\rho_0}\frac{\partial \rho_0}{\partial r} \qquad (10)$$

the term $\propto c_s^2$ is a small correction

- **The equation of hydrostatic equilibrium along the z-coordinate (for $z \ll r$):**

$$\frac{\partial \Phi_0}{\partial z} + \frac{c_s^2}{\rho_0}\frac{\partial \rho_0}{\partial z} = 0 \qquad (11)$$

Equilibrium

- **For the geometrically thin disk ($2h \ll R$), in Eq. (11) one can expand $\partial \Phi_0 / \partial z$ about the orbit plane**

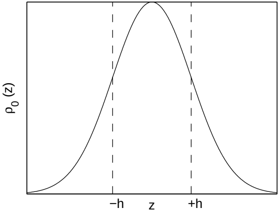
$$\frac{\partial \Phi_0}{\partial z} \approx \mu^2 z \tag{12}$$

$\mu^2 = (\partial^2 \Phi_0 / \partial z^2)|_{z=0}$ **is the frequency of natural vertical oscillations**

- **Equations (11) and (12) then imply**

$$\rho_0(r, z) = \rho_0(r, 0) \exp\left(-\frac{\mu^2 z^2}{2c_s^2}\right) = \rho_0(r, 0) \exp\left(-\frac{z^2}{2h^2}\right) \tag{13}$$

Equilibrium



- **Equilibrium three-dimensional galaxies are inhomogeneous both in r and z directions!**
- **Realistic $\rho_0 = \rho_0(z)$ galaxies are difficult to treat $\Rightarrow$ A simple model: homogeneous in the z-direction slab**
- **The density distribution, which is nearly constant for $|z| \lesssim h$ and rapidly drops beyond $|z| = h$ [Eq. (13)], justifies this approximation**

Perturbation

- **Then Eqs (1)–(6) above are linear in z and, therefore, we may Fourier analyze in the form**

$$X_1(r,\varphi,z,t) = \widetilde{X}\exp(\mathrm{i}qz)\exp(\mathrm{i}k_r r + \mathrm{i}m\varphi - \mathrm{i}\omega t)$$

q is the vertical (along the z-coordinate) wavenumber

- **Inside the layer we have the following equations**

$$\mathrm{i}\omega_* v_r + 2\Omega v_\varphi = \mathrm{i}k_r\Phi_1 + \mathrm{i}k_r c_\perp^2 \frac{\rho_1}{\rho_0} \tag{14}$$

$$\mathrm{i}\omega_* v_\varphi - \frac{\kappa^2}{2\Omega} v_r = \mathrm{i}(m/r)\Phi_1 + \mathrm{i}(m/r) c_\perp^2 \frac{\rho_1}{\rho_0} \tag{15}$$

$$\mathrm{i}\omega_* v_z = \frac{\partial\Phi_1}{\partial z} + \frac{c_z^2}{\rho_0}\frac{\partial\rho_1}{\partial z} \tag{16}$$

PERTURBATION

$$\mathrm{i}\omega_*\rho_1 = \mathrm{i}k_r\rho_0 v_r + \frac{\mathrm{i}m}{r}\rho_0 v_\varphi + \rho_0\frac{\partial v_z}{\partial z} \quad (17)$$

$$\frac{\partial^2\Phi_1}{\partial z^2} - k_\perp^2\Phi_1 \approx 4\pi G\rho_1 \quad (18)$$

- **Next, we match the solutions to the solutions of the Poisson $\nabla^2\Phi_1 = 4\pi G\rho_1$ equation outside the layer**
- **Outside of the layer the solutions of Poisson equation (Laplace equation $\nabla^2\Phi_1 = 0$) are**

$$\Phi_1(z > +h) = +Ce^{-k_\perp(z-h)}e^{\mathrm{i}k_r r + \mathrm{i}m\varphi - \mathrm{i}\omega t} \quad (19)$$

$$\Phi_1(z < -h) = -Ce^{+k_\perp(z+h)}e^{\mathrm{i}k_r r + \mathrm{i}m\varphi - \mathrm{i}\omega t} \quad (20)$$

Generalized dispersion relation

- **In the regions $z > +h$, $z < -h$ and $-h < z < +h$, therefore, a solution may be sought as**

$$\widetilde{\Phi}(z) = \begin{cases} Ae^{-|k_\perp|z} & \textbf{if } z > +h \\ Be^{+iqz} + Ce^{-iqz} & \textbf{if } -h < z < h \\ De^{+|k_\perp|z} & \textbf{if } z < -h \end{cases} \tag{21}$$

- **In Eq. (21), the constants A, B, C, and D are determined by four boundary conditions that both Φ_1 and $\partial\Phi_1/\partial z$ to be continuous at $z = +h$ and $z = -h$**
- **Finally, we have**

$$q \approx \pm\sqrt{\frac{|k_\perp|}{h}} \quad \text{and} \quad k_\perp = \sqrt{k_r^2 + m^2/r^2} \tag{22}$$

Generalized dispersion relation

- **As a result, we arrive at the dispersion relation from which the growth rate $\propto \exp(\Im\omega t)$ is obtained**

$$\omega_*^3 - \omega_*\omega_\mathrm{J} + \omega_\mathrm{grad} = 0 \qquad (23)$$

$\omega_* = \omega - m\Omega$ **is the Doppler-shifted wavefrequency**

- **The standard Lin–Shu dispersion relation (Lin et al. 1969)**

$$\omega_*^2 - \omega_\mathrm{J}' = 0 \qquad (24)$$

- **Our aim is to find both $\Im\omega_* > 0$ and $\Re\omega_*$**
- **Then $X_1 \propto e^{\Im\omega_* t}$ and $\Omega_\mathrm{p} = \Re\omega_*/m$ is the pattern angular speed**

Generalized dispersion relation

- The solution of the **generalized** dispersion relation (23)

$$\omega_{*1,2} \approx s\alpha \left| \kappa^2 - \frac{2\pi G \Sigma_0 (|k_r| + m/r)}{1 + |k_r| h} + (|k_r| + m/r)^2 c_s^2 \right| + \beta m \left| \frac{\partial \Sigma_0}{\partial r} \right|$$

(25)

- The solution of the **standard** dispersion relation (24)

$$\omega_{*1,2} = s\alpha \left| \kappa^2 - 2\pi G \Sigma_0 |k_r| + k_r^2 c_s^2 \right| \quad (26)$$

$m/|k_r| r \ll 1$ **and** $|k_r| h \ll 1$

Also, $s = \mathrm{i}$ **if** $\cdots < 0$ **and** $s = 1$ **if** $\cdots > 0$ **and** $\alpha, \beta = \mathrm{const}$

Astronomical Implications

Disk stability and waves

- **Non-axisymmetric perturbations are more "dangerous" than the axisymmetric ones**
- **In order to suppress the instability of arbitrary but not only circular perturbations: the sound speed (velocity dispersion)**

$$c_s \gtrsim 2c_{\mathrm{T}} \ \text{ or } \ Q \gtrsim 2$$

$c_{\mathrm{T}} = \pi G \Sigma_0 / \kappa$ **is the Safronov–Toomre's sound speed and** $Q = c_s / c_{\mathrm{T}}$ **is the Toomre's stability parameter**

- **The wavelength of the most unstable waves**

$$\lambda \approx 2\lambda_{\mathrm{JT}}$$

$\lambda_{\mathrm{JT}} = c_s^2 / G\Sigma_0$ **is the Jeans–Toomre wavelength**

Disk stability and waves

- **In spiral galaxies**

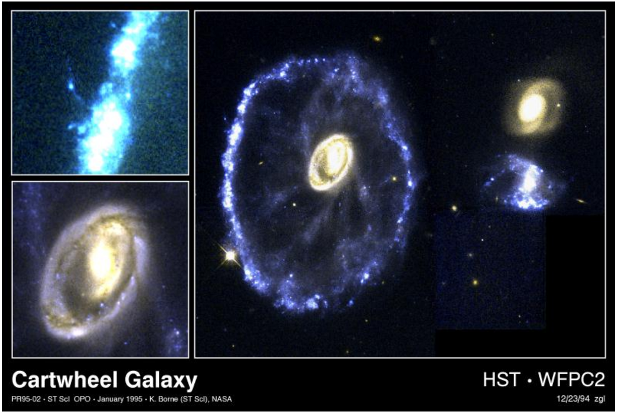
$$\lambda \approx 4\pi h$$

- **Destabilizing self-gravity is far more "dangerous" in thin disks than in thick disks: the effect of finite thickness is a reduction of the growth rate of the gravitational Jeans' instability**
- **The pattern speed of Jeans-unstable spiral perturbations** $\Omega_{\rm p} = \Re\omega_{*1,2}/m$ **does not depend on** m

$$\Omega_{\rm p} = \text{const} \quad \underline{\text{for ALL modes!}}$$

- **The theory states that in spatially homogeneous disks** $\Omega_{\rm p} = 0$

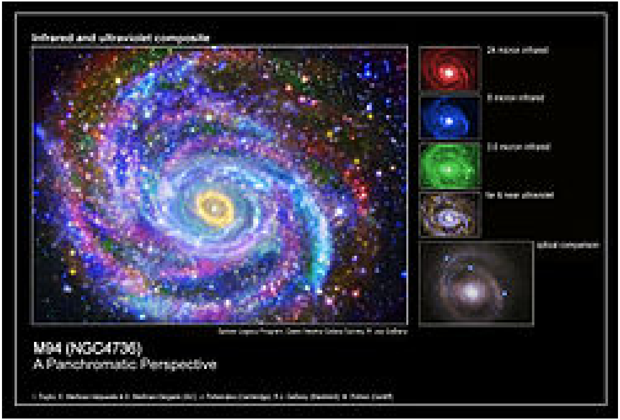
Astronomical Implications



Cartwheel galaxy with two rings and spirals

- **Both** **axisymmetric** **(circular) and** **non-axisymmetric** **(spiral) Lin–Shu density waves?**

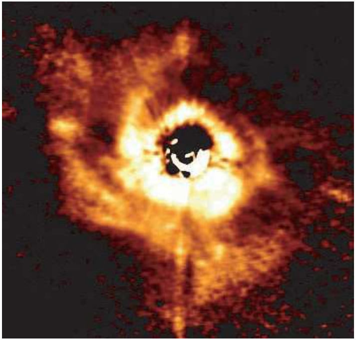
Astronomical Implications



Cartwheel-like galaxy M94 with two rings

- **Both axisymmetric (circular) and non-axisymmetric (spiral) Lin–Shu density waves?**

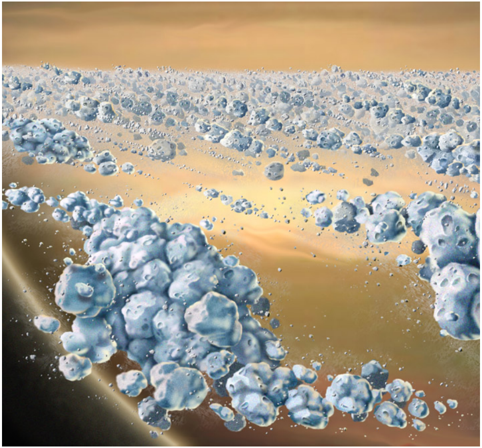
Astronomical Implications



Spiral structure of protoplanetary disks

- **Lin–Shu spiral density waves?**

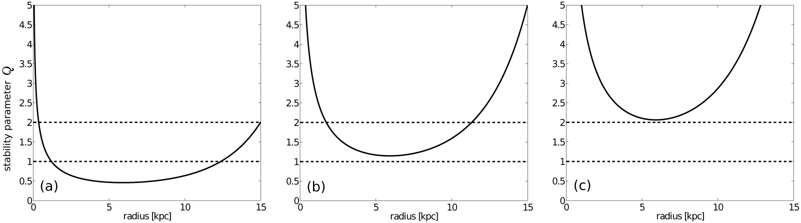
Astronomical Implications



Fine-scale ~ 100 m structure of Saturn's Rings

- **Lin–Shu spiral density waves?**

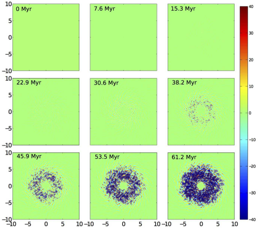
Computer Simulations



Theoretical prediction

- (a) A system is unstable with respect to both **axisymmetric** and **non-axisymmetric** gravity perturbations
- (b) A system is unstable with respect to only **non-axisymmetric** perturbations
- (c) A system is gravitationally Jeans' **stable**!

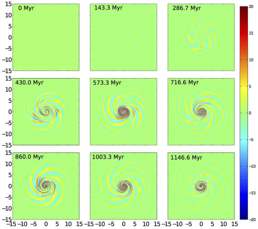
Computer Simulations



Gravitationally Jeans' unstable disk $(Q < 1)$

- **Both axisymmetric and non-axisymmetric unstable density waves**

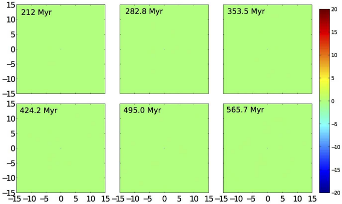
Computer Simulations



GRAVITATIONALLY JEANS' UNSTABLE DISK $(1 < Q < 2)$

- **Only non-axisymmetric unstable density waves**

Computer Simulations



Gravitationally Jeans' stable disk $(Q > 2)$

- **All instabilities are suppressed!**