A new method for reconstructing the density distribution of matter in the disks of spiral galaxies from the rotation velocity curve in it

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Let us consider the axisymmetric model in cylindrical coordinates (ρ, z, φ).

In the disc, we assume that the dust-like matter has stationary rotation (i.e. there is equality between the absolute values of the gravitational and centrifugal forces):

$$\frac{\partial \Phi(\rho, z=0)}{\partial \rho} = \frac{V^2(\rho)}{\rho}. \tag{1}$$

Here, $V(\rho)$ is the rotation velocity curve of the matter and $\Phi(\rho, z)$ is the Newtonian gravitational potential of the dust matter, which can be expressed as

$$\Phi(\rho, z) = \int_0^\infty R\,\mathrm{d}R \int_{-\pi}^{\pi} \mathrm{d}\varphi \frac{-G\sigma(R)}{\sqrt{\rho^2 + R^2 - 2\rho R\cos\varphi + z^2}}. \tag{2}$$

Here, $\sigma(\rho)$ is the surface density distribution of matter in the disc.

The potential $\Phi(\rho, z)$ is a function of $\sigma(\rho)$. Gauss's theorem for this potential leads to the expression:

$$\Phi(\rho, z)_{,z}\,|_{z=0} = 2\pi G\sigma(\rho) \tag{3}$$

Here G is the gravitational constant.

Old method for reconstructing the density distribution was proposed by Alar Toomre (Ap.J. 138, 385, 1963):

$$\sigma(\rho) = \int_0^\infty dk\, J_0(k\rho)\tilde{\sigma}(k)\,, \quad \tilde{\sigma}(k) \equiv \frac{k}{2\pi G}\int_0^\infty dR\, J_1(kR)\, V^2(R) \tag{4}$$

Here $J_0(x)$ and $J_1(x)$ are Bessel functions.

Using this method to determine the surface density $\sigma(\rho)$ has significant shortcomings:

1. To find the desired function $\sigma(\rho)$ it is necessary to integrate a fourfold integral, whereas the Bessel functions themselves are each expressed in terms of integrals.

2. In the integrand of (4) are oscillating and slowly decaying (as $\cos x/\sqrt{x}$) Bessel function, which also makes it difficult to integrate.

3. The integrals has infinite upper limits and therefore they requires a large numerical resources (change of variables leads to new difficulties due to the oscillating integrands).

Jalocha, Bratek & Kutschera (2010) have suggested a more convenient method based on integrals of elliptic functions:

$$\sigma(\rho) = \frac{1}{\pi^2 G} \mathcal{P} \left[\int_0^{\rho} v^2(\chi) \left(\frac{K\left(\frac{\chi}{\rho}\right)}{\rho\,\chi} - \frac{\rho}{\chi} \frac{E\left(\frac{\chi}{\rho}\right)}{\rho^2 - \chi^2} \right) \mathrm{d}\chi + \int_{\rho}^{\infty} v^2(\chi) \frac{E\left(\frac{\rho}{\chi}\right)}{\chi^2 - \rho^2} \,\mathrm{d}\chi \right]$$

Ignorance of the distribution rotation velocity curve *V(ρ)* at large values of *ρ.*

Polyachenko *(2004,* 2005) has studied the same problem (stability of the numerical calculation of *σ).*

We propose a new integral representation for solving the problem of determining the surface density σ(ρ) from the disc velocity curve V(ρ). Our method does not require knowledge of the numerical values of special functions (Bessel functions and/or elliptic functions).

THE NEW METHOD

Using the relation $J_1(x) = -J_0(x)_{,x}$ for the Bessel functions and applying integration by parts to the

integral for $\tilde{\sigma}(k)$ we rewrite expression

$$\sigma(\rho) = \int_0^\infty dk\, J_0(k\rho)\tilde{\sigma}(k)\,, \quad \tilde{\sigma}(k) \equiv \frac{k}{2\pi G}\int_0^\infty dR\, J_1(kR)\, V^2(R)$$

As: $$2\pi G\sigma(\rho) = \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}R\ J_0(k\rho) J_0(kR) \, [V^2(R)]_{,R} \,.$$

Here, we have taken into account that

$$-\int_0^\infty \mathrm{d}R \, \frac{1}{k} J_0(kR)_{,R} \, V^2(R)$$

$$= -\frac{1}{k} J_0(kR) V^2(R) \Big|_0^\infty + \frac{1}{k} \int_0^\infty \mathrm{d}R \, J_0(kR) [V^2(R)]_{,R} \,.$$

Here, we assume that $k \neq 0$. We have also taken into account that at $x = 0$ the function $J_0(x)$ is limited and $V^2(x)$ vanishes (no velocity on the axis). At $x \to \infty$, the opposite is the case; that is, the function $V^2(x)$ is limited and the function $J_0(x)$ fall off as $\cos(x - \pi/4)\sqrt{2/(\pi x)}$.

We denote the gradient of the square of the velocity as $f(R)$:

$$f(R) \equiv [V^2(R)]_{,R}$$

Substituting to the integral the definition of Bessel functions of zero order:

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \mathrm{e}^{\mathrm{i}x \cos \alpha} \, \mathrm{d}\alpha,$$

we gives for the sigma:

$$2\pi G\sigma(\rho) = \int_0^{\pi} \mathrm{d}\alpha \int_0^{\pi} \mathrm{d}\beta \int_{-\infty}^{\infty} \mathrm{d}k \int_0^{\infty} \mathrm{d}R \times \mathrm{e}^{\mathrm{i}k\rho \cos \alpha} \mathrm{e}^{\mathrm{i}kR \cos \beta} \frac{f(R)}{2\pi^2}$$

Here, we have taken into account the fact that $J_0(x)$ is an even function. Thus, the integration over k in equation (8) can be done by changing the limits of integration to go from $-\infty$ to ∞ and by multiplying the result by a factor of 1/2.

Let us now consider the integral representation of the Dirac delta function,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}k,$$ and substitute it in to equation

$$2\pi G\sigma(\rho) = \int_0^{\pi}\mathrm{d}\alpha\int_0^{\pi}\mathrm{d}\beta\int_0^{\infty}\mathrm{d}R\,\delta(\rho\cos\alpha + R\cos\beta)\,\frac{f(R)}{\pi}$$

Here, we take into account the integration rules of the complicated delta function,

$$\delta[g(R)] = \frac{\delta(R-R_0)}{|g(R)_{,R}|},$$

where R_0 are the simple roots of the equation $g(R_0) = 0$.

with respect to α and β reduces to the following two sectors:

$$1. \begin{cases} \alpha \in (0;\ \pi/2) \\ \beta \in (\pi/2;\ \pi) \end{cases}; \qquad 2. \begin{cases} \alpha \in (\pi/2;\ \pi) \\ \beta \in (0;\ \pi/2) \end{cases}.$$

By renaming the variables α and β in these sectors, we can reduce the integral for sigma to the following form:

$$2\pi G\sigma(\rho) = \frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\alpha \int_0^{\pi/2} \mathrm{d}\beta \frac{f(R_0)}{\sin\beta}, \quad R_0 = \rho\frac{\cos\alpha}{\sin\beta}. \qquad (14)$$

We assume that for, $\beta \to 0$ in equation (14), the function f tends to zero fast enough.

This expression (14) is the desired result

Using

$$V^2(R_0),_\rho = V^2(R_0),_{R_0}\, R_0,_\rho = f(R_0)\frac{\cos\alpha}{\sin\beta},$$

we can rewrite equation (14) in the following form:

$$2\pi G\sigma(\rho) = \frac{\mathrm{d}U(\rho)}{\mathrm{d}\rho},$$

$$U(\rho) \equiv \frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\alpha \int_0^{\pi/2} \mathrm{d}\beta \frac{V^2(R_0)}{\cos\alpha} = \int_0^\rho 2\pi G\sigma(\rho)\,\mathrm{d}\rho. \quad (15)$$

Using this expression, it is easy to obtain an expression for the total mass of matter inside the disc radius ρ by integrating equation (15) by parts:

$$GM(\rho) = \int_0^\rho 2\pi G\rho\sigma(\rho)\,\mathrm{d}\rho = U(\rho)\rho - \int_0^\rho U(\rho)\,\mathrm{d}\rho. \quad (16)$$

CHECKING THE METHOD

For basic testing of our new integral representation, we use the following approach. We integrate the expression (1) with respect to ρ:

$$H_1(\rho) \equiv \int_0^{\rho} \frac{V^2(\rho)}{\rho}\, \mathrm{d}\rho. \tag{17}$$

However, an analogous expression can be obtained from equation (2) as

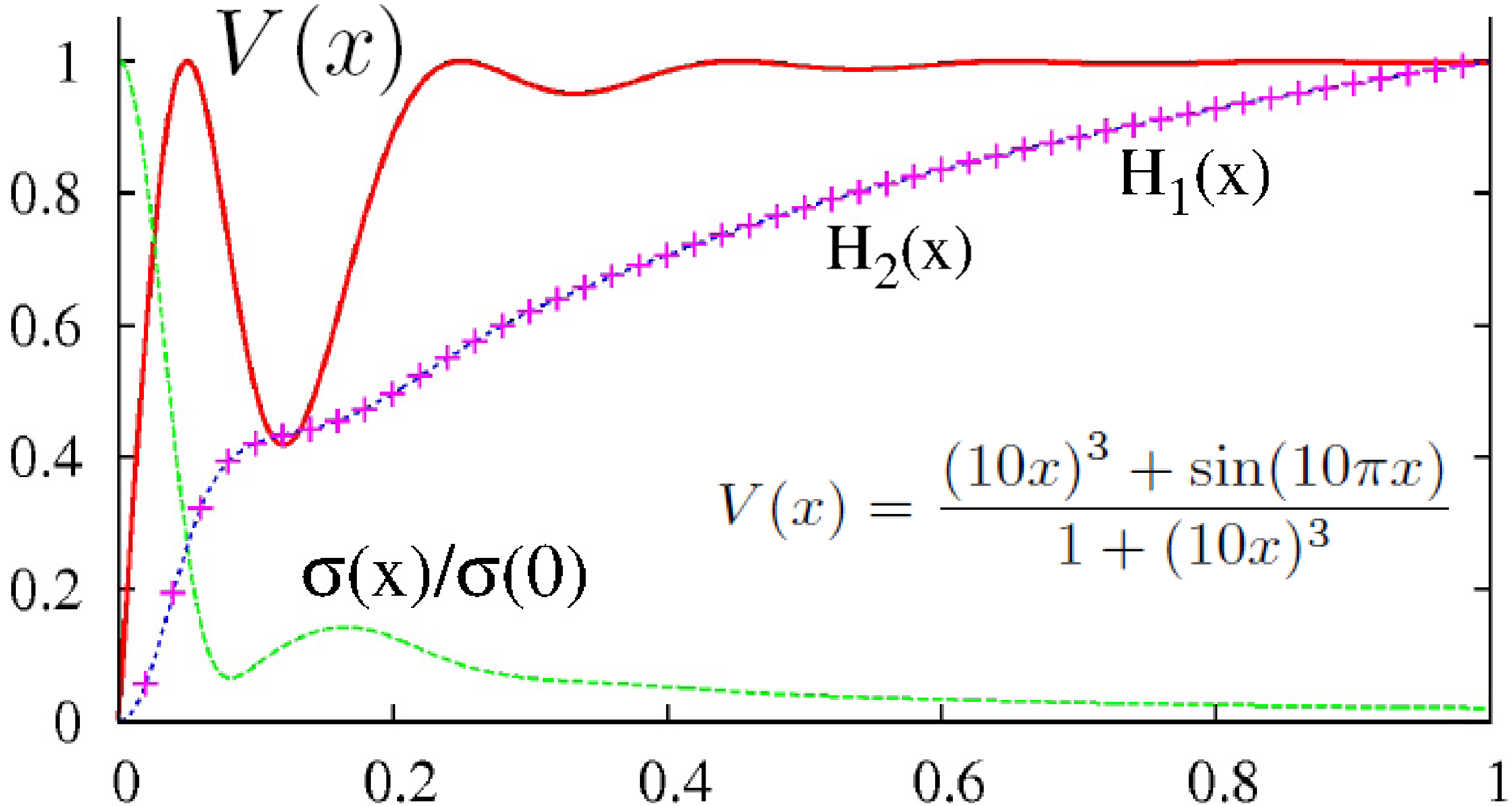
$$H_2(\rho) = U(\infty) - \frac{2}{\pi} \int_0^{\infty} \mathrm{d}R\, 2\pi G \sigma(R) \int_0^{\pi/2} \mathrm{d}\gamma \times \frac{R}{\sqrt{(R-\rho)^2 + 4R\rho \sin^2\gamma}}, \qquad \gamma \equiv \varphi/2 \tag{18}$$

or as

$$H_2(\rho) = \frac{2}{\pi} \int_0^{\infty} \mathrm{d}R\, 2\pi G \sigma(R) \int_0^{\pi/2} \mathrm{d}\gamma \times \left[1 - \frac{R}{\sqrt{(R-\rho)^2 + 4R\rho \sin^2\gamma}}\right]. \tag{19}$$

Expressions for the functions $H_1(\rho)$ and $H_2(\rho)$ ought to be identical, as they must be equal to the expression $\Phi(\rho, z = 0) - \Phi(0, z = 0)$. Thus, we may judge the practical precision of our new method by evaluating and comparing the expressions for $H_1(\rho)$ and $H_2(\rho)$ (using our new method to calculate $\sigma(R)$).

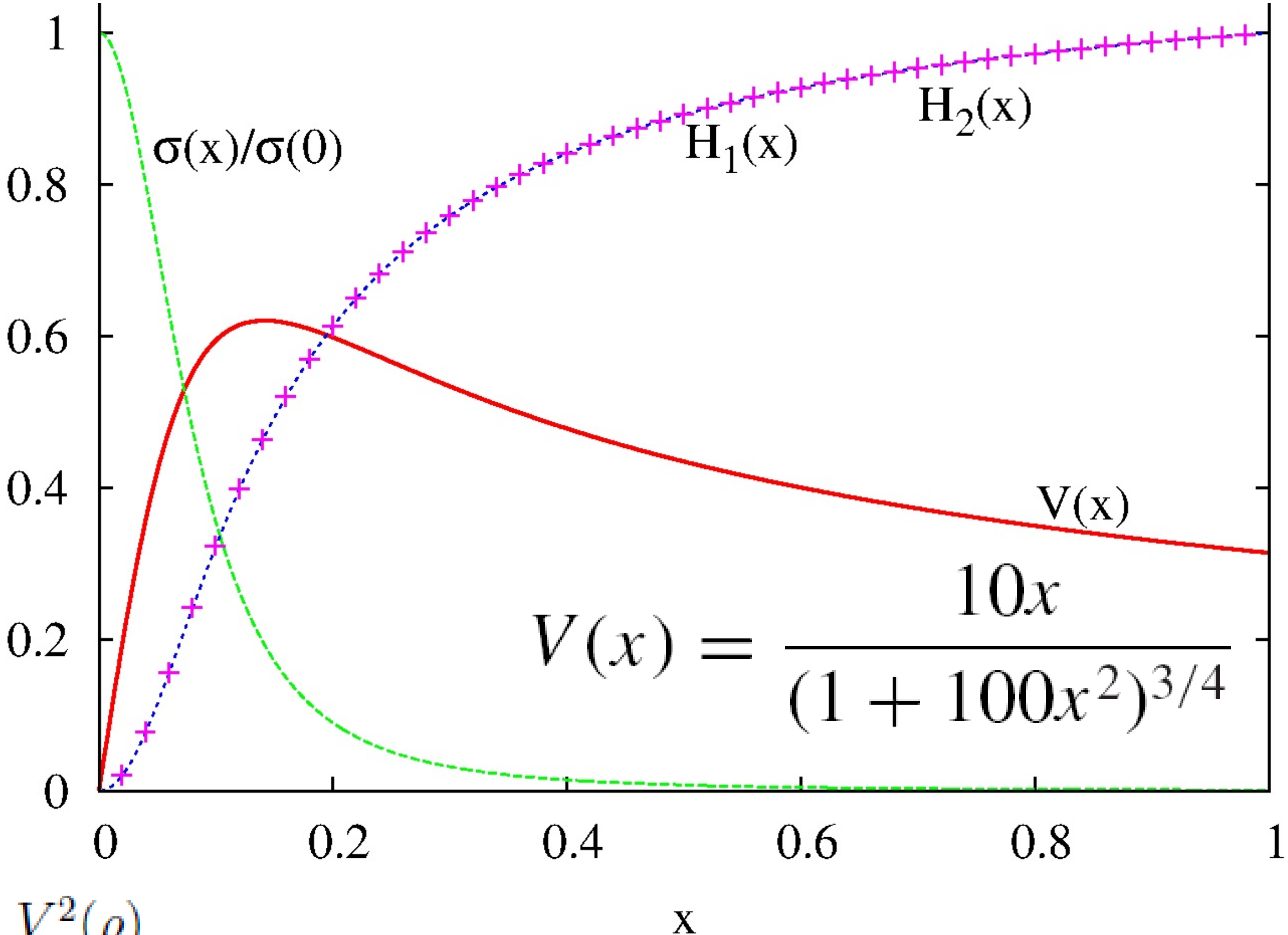
If V(R) corresponds to a constant tail at infinity:



$$H_1(\rho) \equiv \int_0^{\rho} \frac{V^2(\rho)}{\rho}\, d\rho$$

$$H_2(\rho) = \frac{2}{\pi} \int_0^{\infty} dR \cdot 2\pi G \sigma(R) \int_0^{\pi/2} d\gamma \left(1 - \frac{R}{\sqrt{(R-\rho)^2 + 4R\rho \sin^2\gamma}} \right)$$

If V(R) corresponds to a Keplerian tail at infinity:



$$H_1(\rho) \equiv \int_0^{\rho} \frac{V^2(\rho)}{\rho}\, d\rho$$

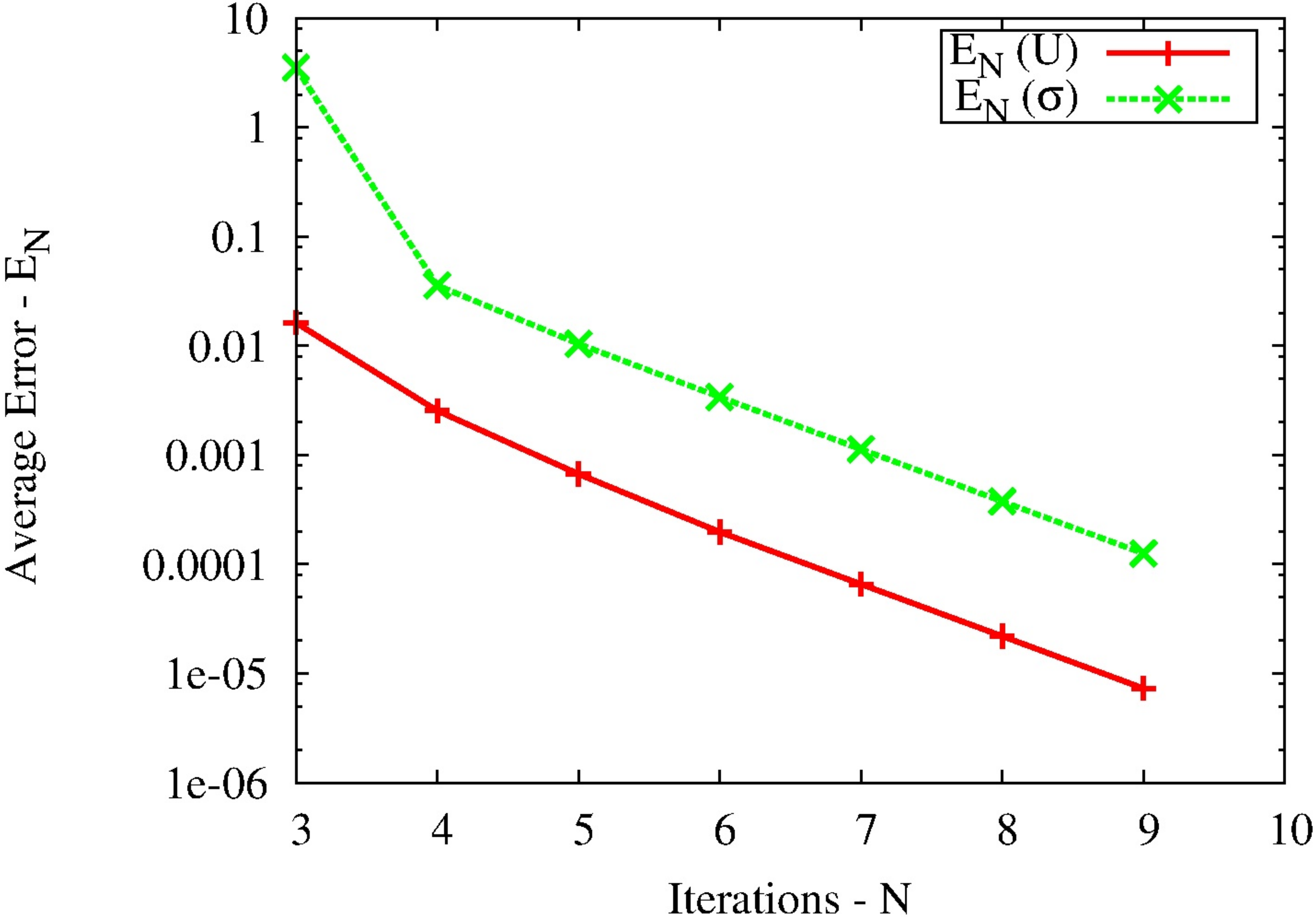
$$H_2(\rho) = \frac{2}{\pi} \int_0^{\infty} dR \cdot 2\pi G\sigma(R) \int_0^{\pi/2} d\gamma \left(1 - \frac{R}{\sqrt{(R-\rho)^2 + 4R\rho \sin^2\gamma}}\right)$$

In the figure, this velocity function and $\sigma(x)$ are calculated by using our new integral representation. The two functions $H_1(x)$ and $H_2(x)$, which we wish to compare, are also plotted.

As can be seen from Fig. 1 (and as is confirmed by detailed inspection of the figure), the two functions $H_1(x)$ and $H_2(x)$ agree very closely. This is also the case for other sample velocity functions that we have tested. Thus, we conclude that our new integral representation is both valid and accurate, even when performing the integrations numerically.

Note that in the expression we used the so-called holographic principle, i.e. each point of the density distribution contains the information about all points of distribution f(R).

Demonstration of convergence test for basic functions of our method:



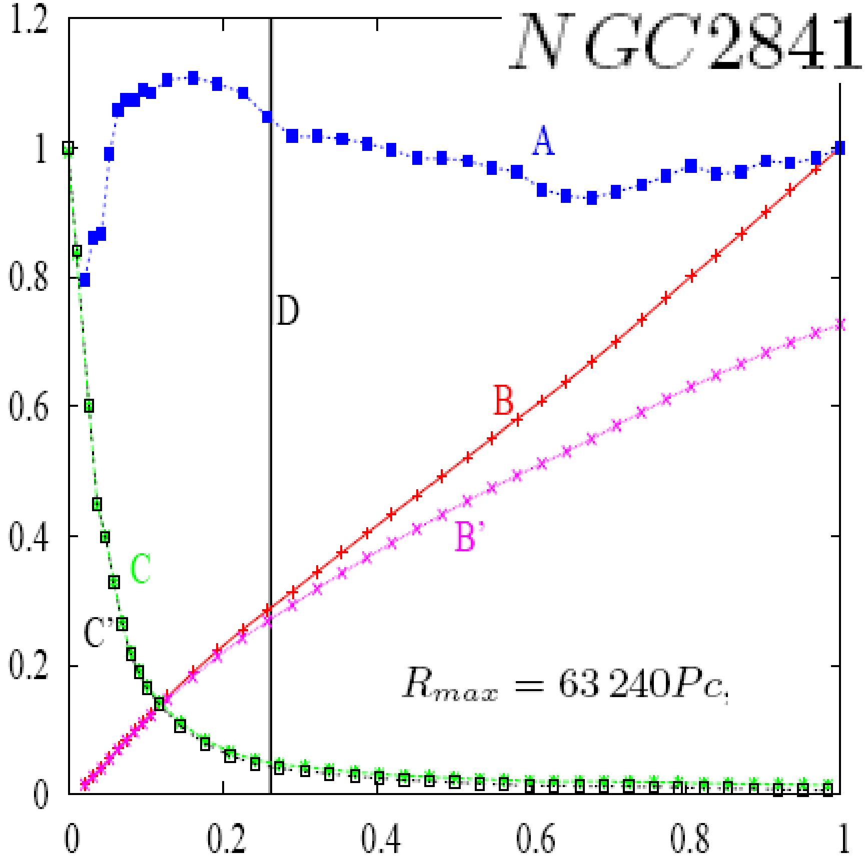
OBSERVATIONAL DATA

By using the method described above, we have restored the matter surface density profiles for the discs of four spiral galaxies, based on their observed rotation curves. The four galaxies are: NGC 2841, 7217, 7331 and 5533.

To compute the behaviour of the curve V(R) at large R>Rmax (where the velocity distribution is unknown), we use the following two cases:

$$\text{(i)} \quad V = V(R_{\max}) = \text{const};$$
$$\text{(ii)} \quad V = V(R_{\max})\sqrt{R_{\max}/R}.$$

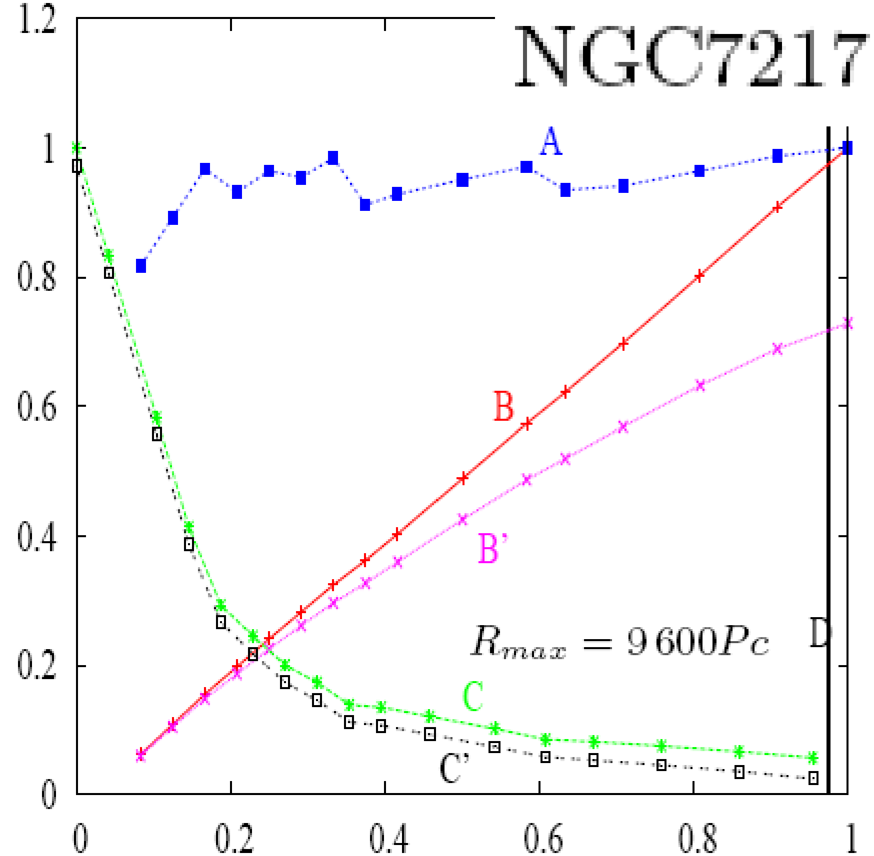
(constant tail and Keplerian tail)



$A)\ V(1) \approx 294 km/sec,$

$B)\ M_{max} = M(1) \approx 1.22 \cdot 10^{12} M_\odot,$

$C)\ \sigma_{max} = \sigma(0) \approx 3\,432 M_\odot/Pc^2,$



$A)\ V(1) \approx 305 km/sec,$

$B)\ M_{max} = M(1) \approx 1.94 \cdot 10^{11} M_\odot,$

$C)\ \sigma_{max} = \sigma(0) \approx 6\,220 M_\odot/Pc^2,$

$C')\ \sigma(0) \approx 6\,050 M_\odot/Pc^2.$

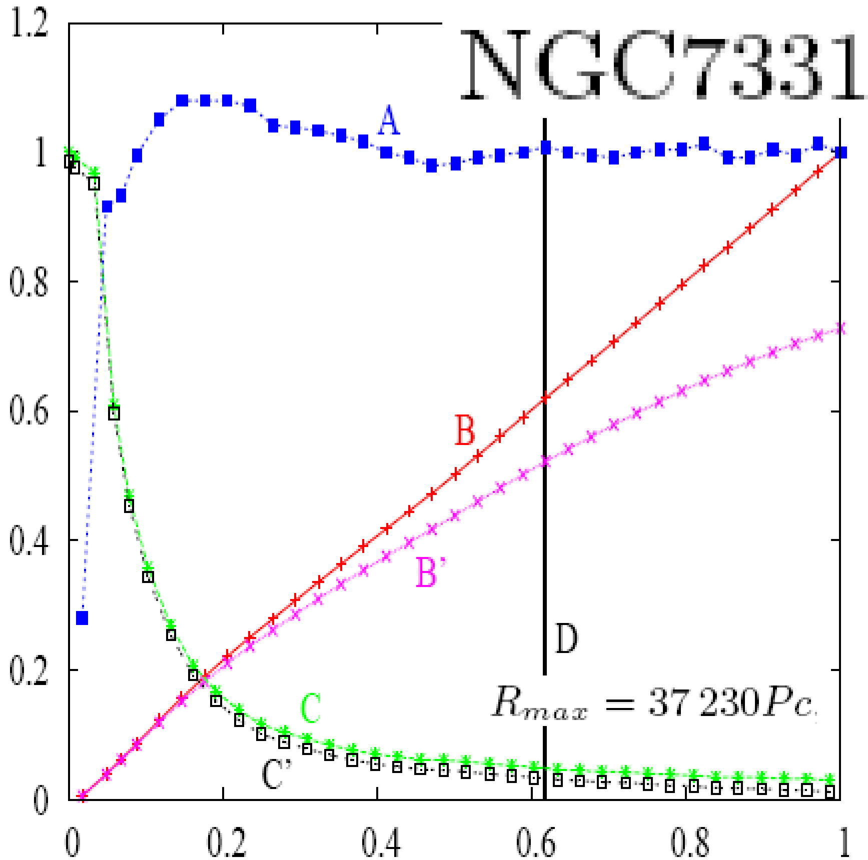
The horizontal axis is the distance in units of x $(x \equiv R/R_{max})$. On the vertical axis plots:

$A)$ – $V(x)/V(1)$;

$B)$ – $M(x)/M_{max}$; $C)$ – $\sigma(x)/\sigma_{max}$ – for $V(R > R_{max}) = V(R_{max}) = const$;

$B')$ – $M(x)/M_{max}$; $C')$ – $\sigma(x)/\sigma_{max}$ – for $V(R > R_{max}) = V(R_{max})\sqrt{R_{max}/R}$;

$D)$ – vertical line marking the end of the stellar disk to the 25-th blue isophotes.

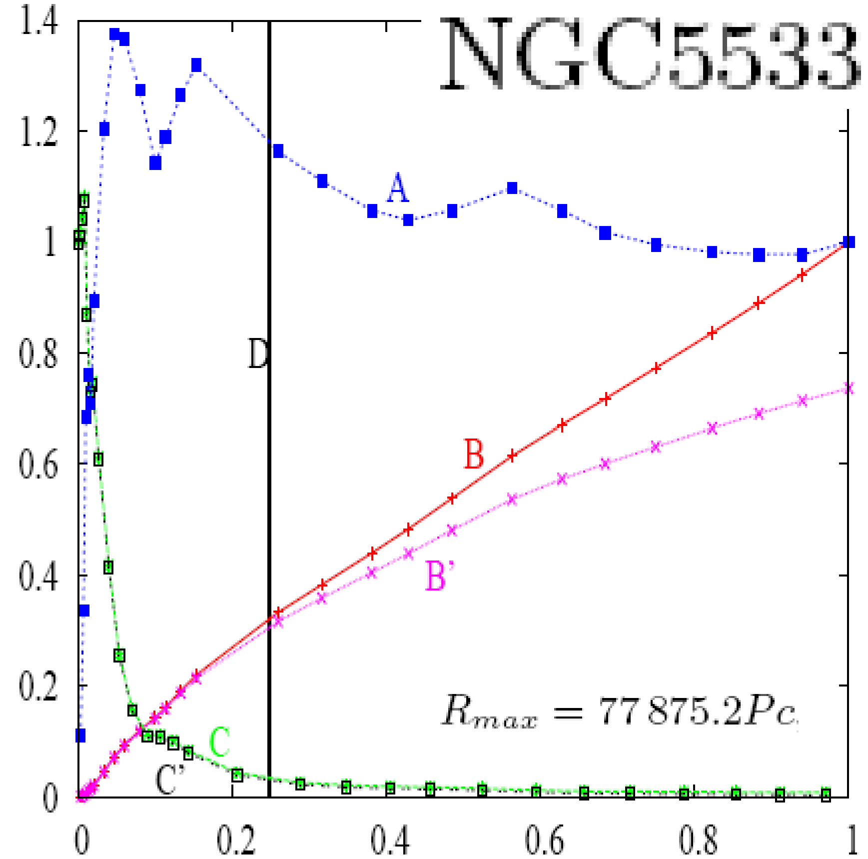


$A)\ V(1) \approx 238 km/sec,$

$B)\ M_{max} = M(1) \approx 4.93 \cdot 10^{11} M_{\odot},$

$C)\ \sigma_{max} = \sigma(0) \approx 1\,883 M_{\odot}/Pc^2,$

$C')\ \sigma(0) \approx 1\,854 M_{\odot}/Pc^2.$



$A)\ V(1) \approx 226 km/sec,$

$B)\ M_{max} = M(1) \approx 9.34 \cdot 10^{11} M_{\odot},$

$C)\ \sigma_{max} = \sigma(0) \approx 2\,877 M_{\odot}/Pc^2,$

$C')\ \sigma(0) \approx 2\,867 M_{\odot}/Pc^2.$

The horizontal axis is the distance in units of x ($x \equiv R/R_{max}$). On the vertical axis plots:

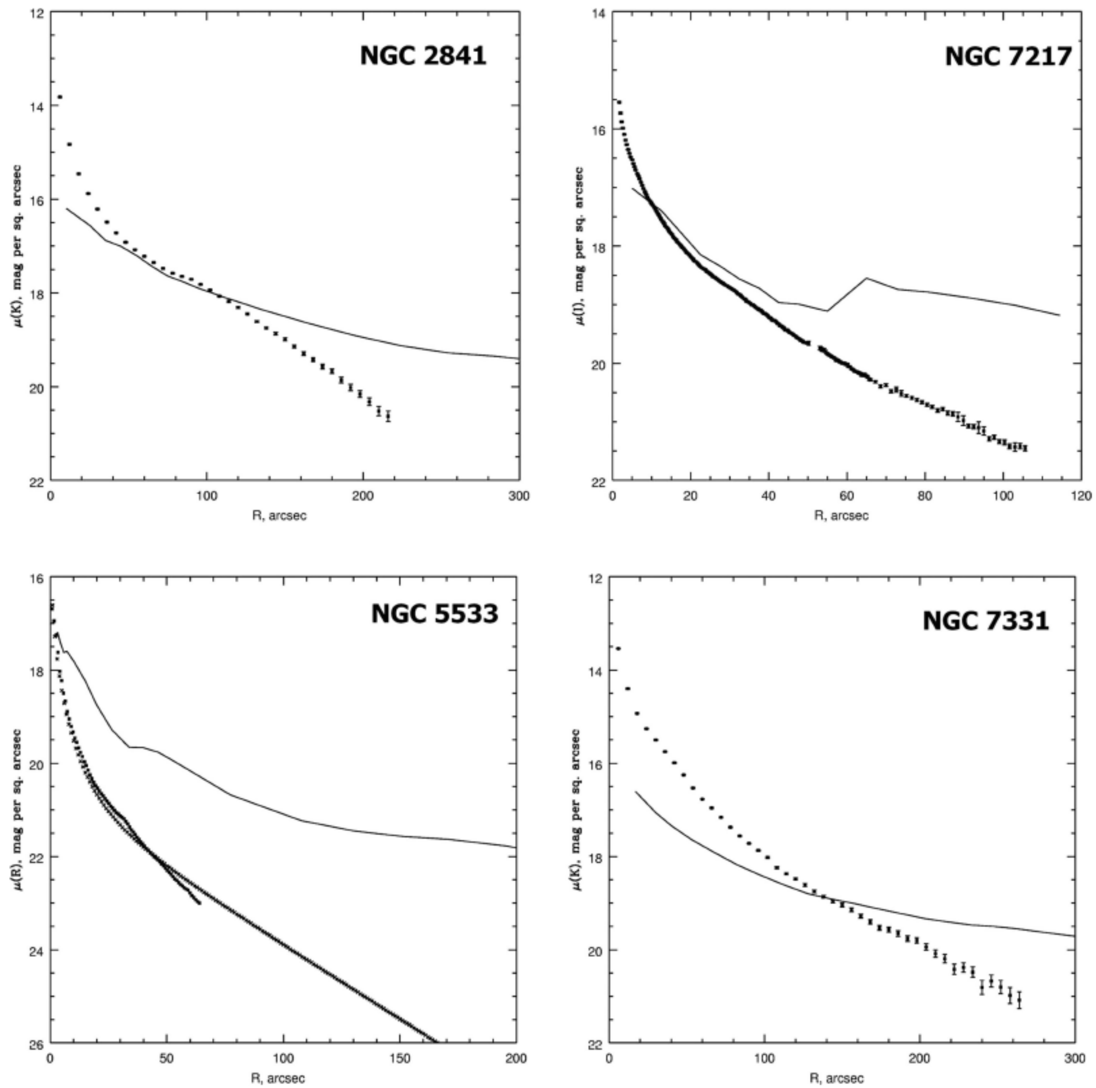
$A)$ – $V(x)/V(1)$;

$B)$ – $M(x)/M_{max}$; $C)$ – $\sigma(x)/\sigma_{max}$ – for $V(R > R_{max}) = V(R_{max}) = const$;

$B')$ – $M(x)/M_{max}$; $C')$ – $\sigma(x)/\sigma_{max}$ – for $V(R > R_{max}) = V(R_{max})\sqrt{R_{max}/R}$;

$D)$ – vertical line marking the end of the stellar disk to the 25-th blue isophotes.

The comparison of the observed surface brightness profiles for our four galaxies, with those based on the gravitating matter surface density. Profiles restored from the rotation curves for the outer rotation curve asymptotic taken as a constant value, calculated by assuming that all the gravitating matter is the stellar component of the galactic discs.



ACCOUNTING FOR THE COMPONENTS OF A SPHERICAL HALO OF THE GALAXY

In all previous considerations, we have ignored the fact that much of the matter of the galaxy, in principle, can be concentrated in a spherically symmetric halo, which has a mass function Msphere(r). Initially, we do not know the distribution of the spherical mass Msphere(r).
This feature is a free parameter of the model. Because of the additivity of the potential (in the Newtonian approximation), it can be divided into two parts (the disc and a spherical component):

$$\Phi(\rho, z) = \Phi_{\rm disc}(\rho, z) + \Phi_{\rm sphere}(r); \qquad r^2 = \rho^2 + z^2.$$

$$\frac{\partial \Phi_{\rm disc}(\rho, z = 0)}{\partial \rho} = \frac{V^2(\rho) - G M_{\rm sphere}(\rho)/\rho}{\rho}$$

Hence, all subsequent arguments amount to replacing the observed features V(ρ) as:

$$V(\rho) \rightarrow \sqrt{V^2(\rho) - GM_{\text{sphere}}(\rho)/\rho}$$

After that, the density of matter in the galaxy can be found by serial iterations
(until observable brightness and velocity curves coincide with it’s theoretical curves).

We have not performed such analysis yet.

CONCLUSIONS

In this work, we have considered a new integral representation for reconstructing the matter surface density in the flat discs of spiral galaxies. Our method (equations 14 and 15) contains only a twofold integral in the restricted limits (from 0 to $\pi/2$).

At first, we suppose that all matter is concentrated in the disc of the galaxy, and we apply the method for reconstructing the surface density of matter in four spiral galaxies (NGC 2841, 7217, 7331 and 5533).

We have compared our results with the distributions of stellar matter. This comparison allows us to make some conclusions about the possible nature of the ‘hidden mass’ in these galaxies. Because the radial distributions of the ‘hidden mass’ that we have derived correlate with the radial neutral hydrogen distributions, this is an argument in favour of the hypothesis that the hidden mass phenomenon at galaxy scales is connected with the cold gas content.

- Thanks !