

Black Hole Spacetimes

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1 Geodesics in Schwarzschild Spacetime

Only months after the publication of General Relativity, Karl Schwarzschild discovered an exactly solution to the Einstein's equation. It describes a static and spherically symmetric vacuum spacetime. In its most well known form, a differential spacetime interval is written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (1)$$

where M is the gravitational mass of the source. Later it was shown by Birkoff that even the Schwarzschild metric (1) is static, it can describe the exterior of any spherically symmetric gravitational source. The star may be collapsing or it may be pulsating, but as long as it remains spherical, the exterior spacetime is described by the Schwarzschild metric (more in Patrick's talk). The location $r = 2M$ is called the Schwarzschild radius. If all the gravitating material is inside the Schwarzschild radius, then the metric (1) describes a black hole, and the Schwarzschild radius is call the event horizon.

To study the geodesics in this spacetime, let us consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\frac{1}{2}\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + \frac{1}{2}r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2), \quad (2)$$

where a dot represents derivative with respect to an affine parameter defined as the ratio of the proper time and the mass of the test particle, $\lambda \equiv \tau/m$. By using λ to parameterize the the geodesic, we can treat both the massive and massless particles on equal footings. Notice that since geodesics are basically free particle motion in a curved spacetime, there is no so-called "gravitational potential", and the Lagrangian above is related to the traditional one by a simply scaling $\mathcal{L} = m(T - V)$ with $V = 0$. All the curvature information is encoded in the way we take inner product of two vectors.

In the Lagrangian (2), t and ϕ are cyclic, corresponding to the conserved energy and \hat{z} component of the angular momentum.

$$E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1 - \frac{2M}{r}\right)\dot{t}, \quad L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \sin^2 \theta \dot{\phi}. \quad (3)$$

The θ equation reads

$$\frac{d}{d\lambda}(r^2 \dot{\theta}) = r^2 \sin \theta \cos \theta \dot{\phi}^2. \quad (4)$$

If we exploit the spherical symmetry of the spacetime to define the initial condition so that $\theta_0 = \pi/2$ and $p^\theta = 0$, then we see that $\theta = \pi/2$ is one solution. By the uniqueness theorem, it is the only solution. This is actually the consequence of total angular momentum conservation. In a spherically symmetric spacetime, the total angular momentum of the test particle is a well defined vector, which means its orbit is confined

in a single plane. Instead of using the Euler-Lagrange equation, we can obtain the r equation from the invariance of rest mass.

$$g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\phi\phi}\dot{\phi}^2 = -m^2. \quad (5)$$

Without loss of generality, we can set m to 1 for massive particle and 0 for massless particle (this freedom is guaranteed by the equivalence principle). Substitute in the various components, we can solve for \dot{r} ,

$$\dot{r}^2 = E^2 - V_{\text{eff}} = E^2 - \left(m^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right). \quad (6)$$

1.1 Time-like Geodesics

When the particle is massive, we set $m^2 = 1$, and the effective potential reads

$$V_{\text{eff}} = \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right). \quad (7)$$

The difference between a constant E^2 and V_{eff} gives the kinetic energy of the orbit. At large distances, it approaches the Keplerian form (apart from an additive constant) $-2M/r + L^2/r^2$. However, when $r \rightarrow 0$, the effective potential is dominated by $-2ML^2/r^3$, which diverges to $-\infty$. Intuitively, this behavior suggests that the centrifugal barrier is not impenetrable. Several such effective potentials are plotted in the left panel of Fig 1, corresponding to different values of L^2 . The local extrema of V_{eff} give the location of circular orbits and are given by solving

$$\begin{aligned} V'_{\text{eff}} &= -\frac{2L^2}{r^3}\left(1 - \frac{2M}{r}\right) + \left(1 + \frac{L^2}{r^2}\right)\frac{2M}{r^2} = 0 \\ \Rightarrow L^2(2M - r) + (r^2 + L^2)M &= 0 \\ \Rightarrow r_{\pm} &= \frac{L^2 \pm L\sqrt{L^2 - 12M^2}}{2M}. \end{aligned}$$

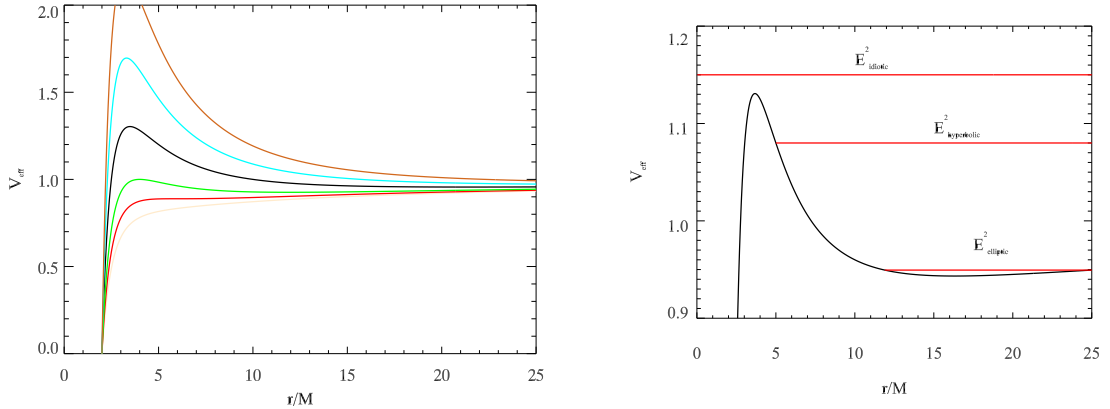


Figure 1: Effective potentials of the Schwarzschild spacetime

For $L^2 > 12M^2$, there are two circular orbits: The outer one is a local minimum, corresponding to a stable orbit, and the inner one is a local maximum, and thus is unstable. When the energy of the orbit is slightly above $V_{\text{eff,min}}$, the particle's orbit oscillate in radius with epicyclic frequency, mimicking the elliptical orbit in Newtonian gravity (labeled by E_{elliptic}^2 in the right panel of Fig 1). However, with the extra $-L^2M/r^3$ term in the effective potential, a relativistic orbit is in general not closed. When the difference between the epicyclic frequency and the Keplerian frequency is small, the orbit can be approximated by a slowly precessing ellipse – a phenomenon first observed in Mercury and served as one of the first experimental tests of General Relativity.

As the particle's energy increases, it changes from bound to unbound. We call them parabolic (for $E^2 = 1$) and hyperbolic (for $E^2 > 1$) even though general relativistic orbits do not strictly follow these curves. However, when E^2 becomes larger than $V_{\text{eff,max}}$, the orbit encounters no turning point, and the particle plunges directly toward the origin. This situation is labeled by E_{idiotic}^2 in the above figure. A moment of thought reviews that for a given value of angular momentum L , increasing energy E corresponds to decreasing the impact parameter. Then it's obvious that the above analysis agrees with our intuition.

When $L^2 = 12M^2$, the two extrema of V_{eff} merge into a single inflection point at $r = 6M$. The circular orbit there is only marginally stable. This radius is called the innermost stable circular orbit (ISCO). For an accretion disk around a Schwarzschild black hole, which consists of a collection of fluid elements approximately following circular orbits, it will have an inner edge at $r = 6M$ if there is no other forces. What's inside the ISCO will have fallen into the black hole along almost radial trajectories.

1.2 Radial Geodesics

While the surface $r = 2M$ is not a physical singularity, it does represent some peculiarity in spacetime. In Schwarzschild coordinates, t is the proper time for an observer at infinity and r is the circumferential radius. They both have intuitive physical meanings. However, inside $r = 2M$, both g_{tt} and g_{rr} change signs. That means r is a time coordinate, and t is now a space coordinate. No observer (massive or massless) can stand still in time. Therefore, if a geodesic (time-like or null) crosses the surface $r = 2M$ from the outside, it must proceed toward ever decreasing r until hitting the curvature singularity at $r = 0$. In particular, if a particles falls into this surface, none of the photons it emits can escape back out; they will necessarily fall into the singularity along with the particle itself. Nothing from $r < 2M$ can communicate with the outside world. For this reason, we call the surface $r = 2M$ the *event horizon*, and call the interior of the event horizon a *black hole*. Of course, a traveler disappears behind the ordinary horizon for very different reasons.

A curious reader might ask the following question. If r is a time coordinate inside the event horizon, why can't it always increase for a geodesic? It is a good question indeed. Since we have only managed to show $\partial_r \cdot \partial_r < 0$, the arrow of "time" is still ambiguous. In fact, if we choose the arrow of time along increasing r , then geodesics can only emerge from the singularity and cross the event horizon from the inside. Nothing from the outside world can communicate with the inside. This spacetime is called a *white hole*. It can be shown that a white hole is highly unstable; it is probably just a mathematical curiosity of little real physical significance. Either way, the event horizon is a one directional membrane.

To understand the strange nature of the event horizon, let us consider purely radial geodesics. Its equation of motion is obtained from the radial equation (6) by setting $L = 0$.

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - 1 + \frac{2M}{r}.$$

The solution can be written parametrically as a cycloid

$$r = \frac{R}{2}(1 + \cos \eta), \quad \tau = \frac{R}{2} \left(\frac{R}{2M}\right)^{1/2} (\eta + \sin \eta),$$

where $R \equiv 2M/(1 - E^2)$ is the radius at which the particle has zero velocity (the ‘‘apastron’’). The initial condition is chosen so that $r = R$ when $\tau = 0$. Notice that there is nothing funny at all at the horizon. The proper time it takes for a particle to fall from R to the origin is

$$\Delta\tau = \frac{\pi}{\sqrt{M}} \left(\frac{R}{2}\right)^{3/2} < \infty.$$

Needless to say, it takes a finite proper time for the particle to reach the event horizon from the outside.

The story is entirely different if we ask how much coordinate time t it takes for a particle to reach the event horizon. In terms of the coordinate time, the radial equation can be rewritten as

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{1 - 2M/r}{E}\right)^2 \left[E^2 - 1 + \frac{2M}{r}\right]. \quad (8)$$

The location $r = 2M$ is a fixed point for nearby solution. To properly analyze the motion near that point, let us consider a first order expansion, with $\epsilon = r - 2M$. After choosing the proper sign for an ingoing geodesic (so that $dr/dt < 0$), we have

$$\frac{d\epsilon}{dt} = -\frac{\epsilon}{2M}.$$

The sign of square root is chosen carefully so that the above equation is valid for either positive or negative values of ϵ . This equation shows that from any point away from the horizon (either inside or outside), it will take an infinite amount of coordinate time for the particle to reach the horizon. If we were allowed to go beyond infinite time, then after crossing the horizon, the particle’s coordinate radius continues to decrease, and its time actually decrease from $t = +\infty$ as well. Eventually, it hits the singularity at $r = 0$ at finite coordinate time. For completeness, we present the solution to the radial equation in coordinate time. It is also written parametrically.

$$t = 2M \log \left| \frac{(R/2M - 1)^{1/2} + \tan(\eta/2)}{(R/2M - 1)^{1/2} - \tan(\eta/2)} \right| + 2M \left(\frac{R}{2M} - 1\right)^{1/2} \left[\eta + \frac{R}{4M}(\eta + \sin \eta) \right],$$

where R is again the apastron, and η the same cycloid parameter. Figure 2 shows several infalling radial geodesics. Since an observer at infinity uses t at his proper time, he will never see any particle fall into the

horizon. Instead, he sees the particle accelerate toward the black hole first and then *decelerates* until coming to a complete stop right on the horizon. This result can be easily inferred from equation (8). In the process, photons emitted from the infalling particle become more and more redshifted. By the time the particle reaches the horizon, even the most sensitive instrument can not detect any photons from it. The particle has disappeared into the black hole, and the event horizon is truly dark.

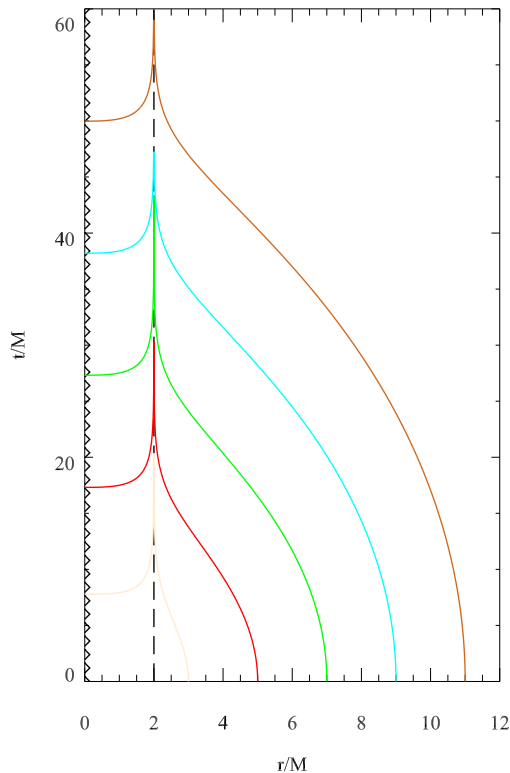


Figure 2: Radially infalling geodesics in Schwarzschild coordinates with different apastrons. All trajectories asymptote to $t \rightarrow \infty$ at $r \rightarrow 2M$, but they reach the singularity at $r = 0$ in finite coordinate time.

1.3 Null Geodesics

When the test particle is massless, the equation of motion is reduced to

$$\left(\frac{dr}{d\lambda}\right)^2 = 1 - \frac{b^2}{r^2} \left(1 - \frac{2M}{r}\right), \quad (9)$$

where $\lambda' = \lambda E$ is the rescaled affine parameter and $b \equiv L/E$ is the impact parameter. There is one unstable circular orbit at $r = 3M$, and the effective potential reaches its maximum value $V_{\text{eff,max}} = b^2/27M^2$ there. If $b < \sqrt{27}M$, the effective potential will never reach 1, which means the orbit will not have a turning point. A distant photon with a small enough value of b will fall directly through the event horizon without escaping. Thus a black hole would cast a shadow of radius $\sqrt{27}M$ over the background source.

2 More General Black Holes

Up to now, we have been studying the simplest kind of black holes described by the Schwarzschild metric. It is spherically symmetric, and static in time. It can be shown that the most general spacetime that admits a *stationary* event horizon is the Kerr-Newman solution. Written in the so-called Boyer-Lindquist coordinates, the metric reads

$$ds^2 = -\frac{\Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \quad (10)$$

where $\rho^2 \equiv r^2 + a^2 \cos^2 \theta$ and $\Delta \equiv r^2 - 2Mr + a^2 + Q^2$.

It represents a black hole that carries electric charge Q and has specific angular momentum a . Notice that this metric is no longer spherically symmetric, nor is it diagonal. More than ever, it is important to remember that these coordinates are nothing more than labels of events in spacetime. Except in specialized circumstances, they do not necessarily have connections to the concept of time and radius that we are used to. In the limit where $Q = 0$, we obtain the Kerr solution, which is an electrically neutral rotating black hole. When $a = 0$, we have the Reissner-Nordström solution which is a charged, but spherically symmetric black hole. When $Q = a = 0$, we recover the good old Schwarzschild solution.

It is surprising that something as complicated as a black hole can be completely determined by only three parameters¹. This uniqueness theorem is summed into a single sentence,

Black holes have no hair.

Any perturbation imparted on the black hole will in general cause the spacetime to become dynamic. However, the time evolution of such spacetime will necessarily lead to another stationary Kerr-Newman spacetime, maybe with different parameters. Higher moments in the perturbation are all carried away by radiation. The parts that are absorbed by the black hole are those incapable of generating radiations (electric monopole, mass monopole, mass dipole). This is the basic idea of the Price theorem, which states that “in a stationary black hole spacetime, anything that can radiate will be radiated away.”

Astrophysically, we don't expect to see too many charged black hole. Even if a black hole was once charged, accretion of matter with opposite charge will very quickly neutralize it. On the other hand, most stars have radii five orders of magnitude larger than their gravitational radii. If we believe that black holes are products of gravitational collapse, and at least a fraction of the angular momentum is conserved in the process, then most (stellar mass) black holes must be very rapidly spinning. Thus the most general black hole relevant to astrophysics is described by the Kerr metric with $Q = 0$ in equation (10).

¹Actually there are four parameters if we include magnetic monopole. However, in classical physics, we usually believe that this parameter is identically zero.

2.1 Dragging of Inertial Frames

The Kerr spacetime is stationary and axisymmetric, and the metric is still independent of these two coordinates. However, in a stationary spacetime, time reversal symmetry $t \rightarrow -t$ is not respected unless it is accompanied by a reversal in the azimuthal angle $\phi \rightarrow -\phi$. Therefore, the two directions $\hat{\mathbf{t}}$ and $\hat{\phi}$ are not orthogonal to each other, and there will be a $g_{t\phi}$ term in the metric. The other two coordinates can be chosen more or less arbitrarily. For convenience, the Boyer-Lindquist coordinates are chosen so that $\hat{\mathbf{r}}$ and $\hat{\theta}$ are orthogonal to the first two basis vectors and to each other. They reduce to the Schwarzschild coordinates in the limit $a \rightarrow 0$.

The presence of a $g_{t\phi}$ term has profound meanings in terms of spacetime structure. Let us understand its significance by exploring geodesics. The symmetries still guarantee two conserved quantities $E = -p_t$ and $L = p_\phi$ along the geodesics. However, the contravariant components of the 4-momentum are now

$$p^t = g^{t\mu} p_\mu = -g^{tt} E + g^{t\phi} L,$$

and

$$p^\phi = g^{\phi\mu} p_\mu = -g^{\phi t} E + g^{\phi\phi} L.$$

A particle launched from infinity with $L = 0$ has impact parameter $b = 0$. In other words, it is aimed directly at the origin. Ordinarily, we expect the particle to fall into the center with no angular motion. However, in the case of a rotating black hole, the particle's trajectory has an angular velocity

$$\omega \equiv \frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{\phi t}}{g^{tt}}.$$

Simple matrix inversion shows

$$g^{tt} = -\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2 \Delta}, \quad g^{\phi t} = -\frac{2aMr}{\rho^2 \Delta}, \quad g^{\phi\phi} = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2 \Delta \sin^2 \theta}.$$

Therefore, the angular velocity of a zero angular momentum is

$$\omega = \frac{2aMr}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}. \quad (11)$$

Notice that this angular velocity decays as r^{-3} , and vanishes when $a = 0$. The phenomenon that a zero angular momentum orbit will acquire angular velocity near the black hole is given the fancy name *dragging of inertial frames*. The particle's angular velocity is not a result of torque, but the spacetime itself. Thus the preferred observer in a Kerr spacetime is rotating with angular velocity ω . This observer is called Zero Angular Momentum Observer (ZAMO), and he carries with him a tetrad given by

$$\begin{aligned} \omega^{(t)} &= |g_{tt} - \omega^2 g_{\phi\phi}|^{1/2} \mathbf{dt}, & \omega^{(r)} &= (\rho/\Delta^{1/2}) \mathbf{dr}, \\ \omega^{(\theta)} &= \rho \mathbf{d\theta}, & \omega^{(\phi)} &= g_{\phi\phi}^{1/2} (\mathbf{d\phi} - \omega \mathbf{dt}). \end{aligned} \quad (12)$$

2.2 Ergosphere

Let us now consider stationary observers in the Kerr spacetime. They might be the physicists who are doing experiments to explore the structure of spacetime. Remember that these observers are not inertial. They need to accelerate to keep their coordinates fixed. We have already seen that in the Schwarzschild spacetime, an observer can not remain at constant radius inside the horizon. In the Kerr geometry, the story is more complicated due to dragging of inertial frames. A stationary observer's 4-velocity is a unit vector that points in the time direction.

$$u^\mu = (u^t, 0, 0, 0), \quad g_{tt} u^{t^2} = -1.$$

However, this condition is not always satisfied. Inside regions where $g_{tt} > 0$, the observer can not remain at constant spatial coordinates. For the Schwarzschild black hole, this occurs inside the horizon $r < 2M$. For a Kerr black hole, the stationary observer's 4-velocity becomes spacelike inside the surface

$$r = M + \sqrt{M^2 - a^2 \cos^2 \theta}. \quad (13)$$

This means no timelike observer can remain at rest with respect to coordinates inside that surface. Very appropriately, this surface is called *the static limit*. The interior of the static limit is called *the ergosphere*.

Another way to understand the meaning of the static limit is to consider photons moving in $\pm\phi$ directions. For null geodesics, we have

$$\begin{aligned} ds^2 = 0 &= g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 \\ \Rightarrow \frac{d\phi}{dt} &= \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} \end{aligned}$$

The minus sign corresponds to prograde orbits (since $g_{t\phi}$ itself is negative) and the plus sign corresponds to retrograde orbits. When $g_{tt} = 0$, the two solutions are

$$\frac{d\phi}{dt} = -\frac{2g_{t\phi}}{g_{\phi\phi}} \text{ and } \frac{d\phi}{dt} = 0.$$

The second solution gives the angular velocity of the retrograde photon. Even the photon is moving at speed of light in the opposite direction of the black hole's rotation, its azimuthal coordinate remains fixed. Anything moving slower must therefore corotate with the black hole. Inside the ergosphere, nothing can resist the inertial frame dragging, and all particles have to move in the same direction as the black hole's rotation.

It is worthwhile to point out that inertial frame dragging and static limit are not unique to Kerr black holes. Any relativistic rotating star will have these properties. If the star is sufficiently compact or sufficiently asymmetric, a portion of its static limit may be outside of the stellar surface. In those cases, the static limit will have the topology of toroids.

2.3 Horizon

Generally speaking, the event horizon is a boundary between the region that can communicate with infinity, and the one that can not. Since causality is established through photons, we expect the event horizon to be a null surface (in other words, every tangent vector to that surface at every point is null). To be more precise, we expect the surface to be generated by *outgoing* photons. While it is beyond the scope of this course to give a rigorous derivation of the horizon for the Kerr black hole, we can offer the answer and give a plausibility argument. Consider any time like observer and the tetrad carried by her. She can try to send photons outward in the direction of ∂_r . In her frame, the photon has 4-momentum

$$\mathbf{P} = E(\mathbf{e}_{(0)} + \mathbf{e}_{(r)}).$$

Therefore, after lapse of some affine parameter, $\delta\lambda$, the radial coordinate change of this photon is

$$\delta r = \delta\lambda \langle \mathbf{dr}, \mathbf{P} \rangle.$$

Keep in mind that \mathbf{dr} is a basis one form, and $\mathbf{e}_{(r)} = (\Delta^{1/2}/\rho)\partial_r$. So if $\delta r > 0$, then the observer can communicate with infinity, and she is outside of the horizon. If $\delta r < 0$, then even the outgoing photons are

moving toward smaller and smaller radii, so she is inside the horizon. The dividing line, given by $\Delta = 0$, is the horizon. For the Kerr metric, it is located at

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \quad (14)$$

When $a = 0$, we recover the Schwarzschild result $r = 2M$. When $a^2 > M^2$, we do not have any horizon in the spacetime. The singularity is naked for the entire universe to see. However, physically, we do not have any means to “spin up” a black hole from $a^2 < M^2$ to $a^2 > M^2$ and thereby destroy the horizon. Any object with sufficiently large angular momentum will scatter off the black hole, rather than fall into it. This is one aspect of the the *cosmic censorship*, which states that one can not produce a naked singularity from any regular data. A Kerr black hole with $a^2 > M^2$ must be created during Big Bang itself.

When $a^2 < M^2$, there are two horizons, which suggest very interesting causal structure for the Kerr spacetime. For our purposes, we shall primarily focus on the outer horizon at $r = r_+$. It is relevant to an outside observer. Notice that the static limit is entirely outside of the horizon.

2.4 Geodesics in Kerr Spacetime

Since the Kerr spacetime is only axisymmetric, geodesics are in general not confined in a single plane like they are in the Schwarzschild spacetime. However, as we can see from the metric (10), the spacetime is reflection symmetric about the equatorial plane defined by $\theta = \pi/2$. If a particle starts out in this plane and with $\dot{\theta} = 0$, then its coordinate value will have $\theta = \pi/2$ for the subsequent motion. This is the only case when geodesics are confined in a single plane, and this is the simplest of all orbits.

Evaluated on the midplane, the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(a^2 + r^2 + \frac{2Ma^2}{r}\right)d\phi^2 - \frac{4aM}{r}dtd\phi + \frac{r^2}{r^2 - 2Mr + a^2}dr^2. \quad (15)$$

As before, stationarity and axisymmetry of the Kerr spacetime guarantee the conservation of energy $E = -p_t$ and angular momentum $L = p_\phi$. With the condition $p^\theta = 0$ and $p_\mu p^\mu = -m^2$, we can still reduce the geodesic equation to a set of first order equations.

$$\begin{aligned} \dot{t} &\equiv p^t = -g^{tt}E + g^{t\phi}L, \\ \dot{\phi} &\equiv p^\phi = -g^{\phi t}E + g^{\phi\phi}L, \\ \dot{r}^2 &\equiv p^{r^2} = g^{rr}[-E^2g^{tt} - L^2g^{\phi\phi} + 2ELg^{t\phi} - m^2], \\ \dot{\theta} &\equiv p^\theta = 0, \end{aligned}$$

where a dot here denotes differentiation with respect to affine parameter. If the test particle is massive, we can set $m = 1$ without losing any generality, since the equivalence principle states that particles of all (nonzero) masses should follow the same geodesic. This is the same as rescaling the affine parameter, and the conserved value of energy and angular momentum. We do, however, need to consider the massless case separately.

Let us now focus on the radial equation. Just as in the Schwarzschild case, this equation determines the qualitative behavior of the orbit. Substituting in the metric coefficients, it reads

$$\begin{aligned} \dot{r}^2 &= E^2 \left[1 + \frac{a^2}{r^2} \left(1 + \frac{2M}{r} \right) \right] - \frac{L^2}{r^2} \left(1 - \frac{2M}{r} \right) - \frac{2aEL}{r^2} \frac{2M}{r} - m^2 \left(1 - \frac{2M}{r} + \frac{a^2}{r^2} \right) \\ &= E^2 + \frac{a^2 E^2 - L^2}{r^2} + \frac{(aE - L)^2}{r^2} \frac{2M}{r} - m^2 \left(1 - \frac{2M}{r} + \frac{a^2}{r^2} \right) \equiv E^2 - W(E, L; r) \end{aligned} \quad (16)$$

One new feature is that the effective potential $W(E, L; r)$ now depends on the energy of the particle. Nevertheless, the analysis remains parallel. Since the right hand side is the square of the radial momentum, it

must be non-negative. Therefore, the regions where $E^2 < W$ are forbidden. If a particle can reach spatial infinity, its radial equation asymptotes to $\dot{r}^2 = E^2 - m^2$, which is the correct flat spacetime expression. For bound orbits, $E^2 < m^2$, and they can be viewed as (nonlinear) oscillations about circular orbits. To have circular orbits, we need to satisfy $\dot{r} = \ddot{r} = 0$, which is equivalent to

$$\begin{aligned} \frac{\partial W}{\partial r} &= \frac{2(a^2 E^2 - L^2)}{r^3} + \frac{6M(aE - L)^2}{r^4} + m^2 \left(\frac{2M}{r^2} - \frac{2a^2}{r^3} \right) = 0 \\ \Rightarrow r(a^2 E^2 - L^2) + 3M(aE - L)^2 + m^2(Mr^2 - a^2 r) &= 0 \\ \Rightarrow r_{\pm} &= \frac{a^2(m^2 - E^2) + L^2 \pm \sqrt{[a^2(m^2 - E^2) + L^2]^2 - 12m^2 M^2 (aE - L)^2}}{2m^2 M}. \end{aligned}$$

Differentiation one more time shows that the circular orbit at r_+ is stable, and the one at r_- is unstable. Of course, these circular orbits can exist only if the argument of the square root (i.e., the discriminant) in the above expression is positive. When it is negative, no circular orbit is possible, and particles plunge directly into the black hole. The last stable circular orbit is obtained when the discriminant vanishes, and it occurs at a radius

$$r_{\text{ISCO}} = \frac{\sqrt{3}|aE - L|}{m}.$$

Notice that in the limit $a \rightarrow 0$, we have $L = \sqrt{12}Mm$, which leads to the familiar result $r_{\text{ISCO}} = 6M$. To obtain an explicit answer, we can eliminate E and L by requiring simultaneously vanishing of the discriminant and $\dot{r} = 0$. The algebra is straight forward but tedious, and the result is

$$r_{\text{ISCO}} = M \left\{ 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \right\}, \quad (17)$$

where

$$Z_1 = 1 + \left(1 - \frac{a^2}{M^2} \right)^{1/3} \left[\left(1 + \frac{a}{M} \right)^{1/3} + \left(1 - \frac{a}{M} \right)^{1/3} \right], \quad Z_2 = \left(3 \frac{a^2}{M^2} + Z_1^2 \right)^{1/2}.$$

The \mp sign corresponds to prograde and retrograde orbits. As we increase the value of a , the radius of the last stable prograde circular orbit decreases from $6M$ to M . The binding energy per unit rest mass at the last stable orbit is given by $1 - E_{\text{ISCO}}/m$, and it increases from $1 - \sqrt{8}/3$ for the Schwarzschild black hole to $1 - 1/\sqrt{3}$ for a maximally rotating Kerr black hole ($a = M$). This is of course due to the fact that particles can orbit closer to the event horizon before plunging in. We need to remind ourselves that although $r_{\text{ISCO}} = r_{\text{EH}}$ for a maximally rotating Kerr black hole, they do not label the same physical location in spacetime. A careful calculation shows that these two radii are separated by a finite proper distance.

A peculiar property of the orbits in Kerr spacetime is the existence of negative energy orbits. To see this, let us write the radial equation (16) as

$$\dot{r}^2 = (E - W_+)(E - W_-),$$

where

$$W_{\pm} = \frac{2aLM \pm \sqrt{r[a^2 + r(r - 2M)][a^2 m^2(2M + r) + r(L^2 + m^2 r^2)]}}{r^3 + a^2(2M + r)}.$$

The curves of W_{\pm} are plotted in Fig 3. The allowed regions are $E > W_+$ and $E < W_-$. However, a moment's thought tells us that orbits with $E < W_-$ are unphysical. Those orbits will imply $E < -m$ in flat spacetime, which is absurd if we want to interpret E as energy. This ambiguity comes from the fact that we only require $E^2 > m^2$, and is associated with the direction we choose to parameterize the geodesics. For now on, let us only consider particles with $E > W_+$. When $L > 0$, W_+ behaves normally, and we see the usual finite centrifugal barrier. However, when $L < 0$, funny things happen for W_+ , and it becomes negative at

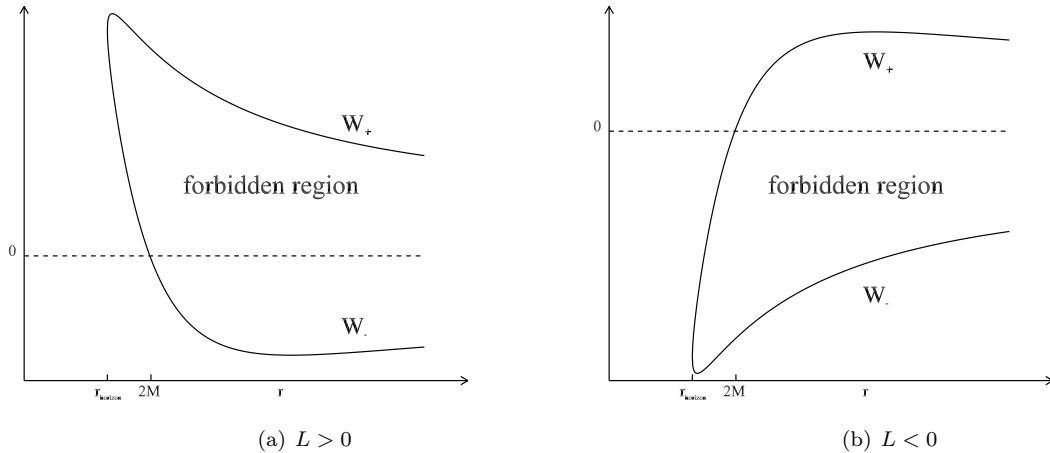


Figure 3: Effective potential for a Kerr black hole with $a = 0.95M$ and $L = 10M$.

some value $r_0 < 2M$. That is, the effective potential becomes negative somewhere inside the ergosphere (but outside of the event horizon). If we were to allow all values of $E > W_+$, we must have some orbits inside the ergosphere with $E < 0$. Obviously they must be trapped by the black hole. The existence of negative energy orbits has profound meaning in energy extraction from the black hole. We shall return to this point shortly.

So far we have been studying orbits in the equatorial plane only. What happens when a particle is originated away from this plane? In general, it will follow complicated geodesic, and $p^\theta \neq 0$. Fortunately, there is another constant of motion, which is related to the angular momentum perpendicular to the black hole's rotation axis. It was first discovered by Carter, and hence is called Carter's constant.

$$Q = p_\theta^2 + \cos^2 \theta \left[a^2 (m^2 - E^2) + \left(\frac{L}{\sin \theta} \right)^2 \right]. \quad (18)$$

Once again, using the three constants of motion, and normalization of the 4-momentum, we can reduce the geodesic equation to a set of first order differential equations.

2.5 Energy Extraction

The existence of an ergosphere outside of the event horizon offers the opportunity to extract energy from a rotating black hole (Ergo is Greek for energy). The mechanism was originally proposed by Penrose. If an orbit extends to outside of the ergosphere, then the conserved quantity associated with stationarity must be interpreted as the energy at infinity $E = -p_t$, and it must be positive. However, $g_{tt} > 0$ inside the ergosphere, which means the direction $\hat{\mathbf{t}}$ is no longer pointing in the time direction. Instead, it becomes spacelike. If an orbit is entirely enclosed by the ergosphere, then the conserved E is simply one spatial component of the momentum, which can and does take on negative values.

Now we consider sending a particle of 4-momentum \mathbf{p} into the ergosphere. It has energy $E \equiv -\mathbf{p} \cdot \partial_t$. It explodes inside the ergosphere into two pieces, with 4-momenta \mathbf{p}_1 and \mathbf{p}_2 . Conservation of 4-momentum requires $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$. Assume the orbit of \mathbf{p}_1 is such that $-\mathbf{p}_1 \cdot \partial_t \equiv E_1 < 0$. As we have seen earlier, this piece will necessarily fall into the black hole. If the other piece manages to escape to infinity, it will have energy $E_2 \equiv -\mathbf{p}_2 \cdot \partial_t = E - E_1$. Since $E_1 < 0$, we see that $E_2 > E$. In other words, energy has been

extracted from the black hole. Where does the energy come from? To answer that question, we consider an observer at a constant radius and polar angle inside the ergosphere. She will have 4-velocity

$$\mathbf{u}_{\text{obs}} = u_{\text{obs}}^t (\partial_t + \Omega_{\text{obs}} \partial_\phi).$$

As far as the observer is concerned, her 4-velocity is still a unit vector pointing in the time direction, even though the “time” is no longer t . The local physics is indistinguishable from special relativity, which means the energy measured by that observer must be non-negative. In particular

$$-(\partial_t + \Omega_{\text{obs}} \partial_\phi) \cdot \mathbf{p}_2 = E_2 - \Omega_{\text{obs}} L_2 \geq 0.$$

Inside the ergosphere, $\Omega_{\text{obs}} \geq 0$. Therefore, if $E_2 < 0$, we must have $L_2 < 0$. As this particle falls into the black hole, it reduces the black hole’s total mass and angular momentum. To a distant observer, the extra energy of particle 2 thus comes from the rotational energy of the black hole.

Unfortunately, the original Penrose process is unlikely to occur in nature. In order for one piece to escape to infinity and the other fall into the black hole, energy exchange must be large, and hence the relative velocity between them should be a significant fraction of unity. This can be understood qualitatively from the effective potential diagram in Fig 3. For a particle to reach the ergosphere, its energy must be $E_{\text{ini}} > \max(W_+)$. However, after the explosion, for one piece to have negative energy, it must lose at least $\Delta E \gtrsim E_{\text{ini}}$. In astrophysics, very few explosion is that powerful. The idea of energy extraction from rotating black holes was resurrected by Blandford and Znajek. Instead of sending real particles, they imagined sending electromagnetic fields. Since electromagnetic disturbances travel at the speed of light, the large orbit separation problem is automatically solved. We shall postpone the detailed discussion of this process and its variants until we develop the formalism for magnetohydrodynamics in curved spacetime.

In the energy extraction process (either by Penrose or Blandford-Znajek method) negative energy is fed into the black hole, thus decreasing its mass M . However, as Hawking pointed out, the total surface area of a black hole, A , can never decrease in any classical process. Motivated by this idea, we can defined the irreducible mass of a black hole as

$$M_{\text{irr}} = \left(\frac{A}{16\pi} \right)^{1/2}.$$

No physical processes can reduce the black hole mass below this value. For a Kerr black hole, its area can be computed as

$$A = 8\pi M \left[M + \sqrt{M^2 - a^2} \right]. \quad (19)$$

We may invert this expression to give the total mass M as a function of the irreducible mass and angular momentum.

$$M^2 = M_{\text{irr}}^2 + \frac{J^2}{4M_{\text{irr}}^2}.$$

The fraction of the black hole mass that can be extracted and transported out is

$$1 - \frac{M_{\text{irr}}}{M} = 1 - \sqrt{\left(1 + \sqrt{1 - a^2/M^2}\right)/2}. \quad (20)$$

Maximum efficiency is achieved when the black hole is maximally rotating with $a = M$. In that case, approximately 29.3% of the initial mass can be used as energy source for external phenomena.

3 Post Newtonian Approximations

In many astrophysical applications (e.g., when the system under consideration is far away from the black hole event horizon), the gravitational field is rather weak, and the essence of Relativity can be captured

by considering only the Newtonian gravity with a small correction. Conceptually, this procedure of adding successive correction to the Newtonian gravity is merely a Taylor series expansion of the full Einstein field equations. To regulate this expansion, we introduce a parameter of smallness $\varepsilon \sim v/c$, where v is the typical velocity in the system. Virial theorem then tells us that the Newtonian gravitational potential is $|\Phi| \sim \varepsilon^2$, the energy density is $\rho \sim \varepsilon^2$ and the pressure is $p \sim \varepsilon^4$.

When there is no gravity in spacetime, spacetime is flat. Any weak gravity field can be viewed as perturbation on the background Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Time reversal symmetry of any gravitational theory implies that h_{00} and h_{ij} only contain even powers of ε , and h_{0i} only contain odd powers of ε . The Newtonian gravity is described geometrically as

$$h_{00} = -2\Phi + O(\varepsilon^4), \quad h_{0i} = O(\varepsilon^3), \quad h_{ij} = O(\varepsilon^2), \quad (21)$$

where Φ is the gravitational potential. Next, we may perform spatial coordinate transformation so that the spatial metric is diagonal (since the spatial metric is represented by a symmetric matrix). If spacetime is isotropic, then we can not tell the difference between different directions. Hence the spatial metric must be proportional to the identity matrix, $h_{ij} = \delta_{ij}U$. If we expand the full Einstein equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ to first order in $h_{\mu\nu}$, the linearized equations take the form

$$-\square \bar{h}^{\mu\nu} = 16\pi T^{\mu\nu}, \quad \bar{h}^{\mu\nu}{}_{,\nu} = 0, \quad \text{where } \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}. \quad (22)$$

By solving the linearized equations, the metric perturbation to order ε^3 is

$$h^{\mu\nu} = \begin{pmatrix} -2\Phi & -A^x & -A^y & -A^z \\ -A^x & -2\Phi & 0 & 0 \\ -A^y & 0 & -2\Phi & 0 \\ -A^z & 0 & 0 & -2\Phi \end{pmatrix}.$$

The trace reversed potential is

$$\bar{h}^{\mu\nu} = \begin{pmatrix} -4\Phi & -A^x & -A^y & -A^z \\ -A^x & 0 & 0 & 0 \\ -A^y & 0 & 0 & 0 \\ -A^z & 0 & 0 & 0 \end{pmatrix}.$$

Written in component form, the gauge condition reads

$$-4\dot{\Phi} - \vec{\nabla} \cdot \vec{A} = 0, \quad \dot{\vec{A}} = 0.$$

If we only need ε^3 terms in the stress-energy tensor, the field equations are

$$\begin{aligned} \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)4\Phi &= 16\pi\rho, \\ \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\vec{A} &= 16\pi\rho\vec{v}. \end{aligned}$$

All the vectors are in flat spacetime. The gauge condition implies $\ddot{\Phi} = 0$, so the 00 component of the field equation just reduce to the Poisson's equation $\nabla^2\Phi = 4\pi\rho$. Next, we use the vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2\vec{A},$$

to convert the spatial component of the field equation to

$$-4\vec{\nabla}\dot{\Phi} - \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = 16\pi\rho\vec{v}.$$

If we define the “gravito-electric” field $\vec{g} = -\vec{\nabla}\Phi$ (which is just the gravitational acceleration), and the “gravito-magnetic” field $\vec{B} = \vec{\nabla} \times \vec{A}$, then the linearized field equations become

$$\begin{aligned}\vec{\nabla} \times \vec{g} &= 0, & \vec{\nabla} \cdot \vec{g} &= -4\pi\rho, \\ \vec{\nabla} \cdot \vec{B} &= 0, & \vec{\nabla} \times \vec{B} &= 4\left(\frac{\partial\vec{g}}{\partial t} - 4\pi\rho\vec{v}\right).\end{aligned}\tag{23}$$

Other than some difference in sign and numerical coefficients, these equations have the same structure as Maxwell equations. Thus we may (carefully) borrow some of the intuition from electromagnetism to understand phenomena under weak gravitational fields.

As an example, let us consider a time-like geodesic in a Kerr spacetime. Far away from the black hole, we have a “gravito-magnetic vector potential” given by

$$A_\phi = \frac{g_{t\phi}}{r \sin\theta} \approx -\frac{2M}{r^2}a \sin\theta,\tag{24}$$

which corresponds to a magnetic dipole with $\mu = 2Ma$. Since the “gravito-electric potential” decays as $2M/r$, the dominant contribution to gravity comes from the spherically symmetry part if we remain far from the black hole. A test particle orbiting in this spacetime is approximately confined in a single plane. Analogous to electromagnetism, the circular motion of the test particle behaves like a “gravito-magnetic dipole”. Thus, unless the orbital plane is perpendicular to the black hole spin axis, this plane will precess at a frequency given approximately by $\omega \approx 2Mar^{-3}$. This phenomenon is known as the Lense-Thirring effect.

Accretion disks around black holes may be thought of as a collection of fluid elements in such quasi-circular orbits with very small radial drift velocities. Depending on the fueling mechanism of the disk (e.g., from a companion star in an X-ray binary or from the large galactic scale in an AGN), the outer rim of the accretion disk may not necessarily be perpendicular to the black hole spin axis. In that case, each annulus will precess at its corresponding frequency about the equatorial plane of the black hole. In general, viscous effects will damp the amplitude of oscillation as gas drifts inward slowly. Typically, the inner region of the disk ($\sim 500M$) is completely relaxed into the equatorial plane.